

L-KURAMOTO-SIVASHINSKY SPDEs IN ONE-TO-THREE DIMENSIONS: L-KS KERNEL, SHARP HÖLDER REGULARITY, AND SWIFT-HOHENBERG LAW EQUIVALENCE

HASSAN ALLOUBA

ABSTRACT. Generalizing the L-Kuramoto-Sivashinsky (L-KS) kernel from our earlier work, we give a novel explicit-kernel formulation useful for a large class of fourth order deterministic, stochastic, linear, and nonlinear PDEs in multi-spatial dimensions. These include pattern formation equations like the Swift-Hohenberg and many other prominent and new PDEs. We first establish existence, uniqueness, and sharp dimension-dependent spatio-temporal Hölder regularity for the canonical (zero drift) L-KS SPDE, driven by white noise on $\{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3$. The spatio-temporal Hölder exponents are exactly the same as the striking ones we proved for our recently introduced Brownian-time Brownian motion (BTBM) stochastic integral equation, associated with time-fractional PDEs. The challenge here is that, unlike the positive BTBM density, the L-KS kernel is the Gaussian average of a modified, highly oscillatory, and complex Schrödinger propagator. We use a combination of harmonic and delicate analysis to get the necessary estimates. Second, attaching order parameters ε_1 to the L-KS spatial operator and ε_2 to the noise term, we show that the dimension-dependent critical ratio $\varepsilon_2/\varepsilon_1^{d/8}$ controls the limiting behavior of the L-KS SPDE, as $\varepsilon_1, \varepsilon_2 \searrow 0$; and we compare this behavior to that of the less regular second order heat SPDEs. Finally, we give a change-of-measure equivalence between the canonical L-KS SPDE and nonlinear L-KS SPDEs. In particular, we prove uniqueness in law for the Swift-Hohenberg and the law equivalence—and hence the same Hölder regularity—of the Swift-Hohenberg SPDE and the canonical L-KS SPDE on compacts in one-to-three dimensions.

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1. INTRODUCTION AND STATEMENTS OF RESULTS

We give a novel, unifying, and very useful explicit-kernel (mild) formulation for a large class of linear, nonlinear, deterministic, and stochastic fourth order PDEs that includes many new, as well as prominent, equations. We focus in this article on the L-Kuramoto-Sivashinsky (L-KS) stochastic PDEs¹ (SPDEs):

$$(1.1) \quad \begin{cases} \frac{\partial U}{\partial t} = -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 U + b(U) + a(U) \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\ U(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

where $(\varepsilon, \vartheta) \in (0, \infty) \times \mathbb{R}$ is a pair of parameters, $a, b : \mathbb{R} \rightarrow \mathbb{R}$ and $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ are Borel measurable, and $\partial^{d+1} W / \partial t \partial x$ is the space-time white noise corresponding to the real-valued Brownian sheet² W on $\mathbb{R}_+ \times \mathbb{R}^d$, $d = 1, 2, 3$. In particular $b(u)$ may be a polynomial of (1) Allen-Cahn type $b(u) = \sum_{k=0}^{2p-1} c_k u^k$ for $p \in \mathbb{N}$ and for $c_{2p-1} < 0$, to get many interesting fourth order SPDEs with an Allen-Cahn type nonlinearity (including a generalized Swift-Hohenberg SPDE when $\varepsilon, \vartheta > 0$), or of (2) KPP type $b(u) = \sum_{k=0}^{2p} c_k u^k$ for $p \in \mathbb{N}$ and for $c_{2p} < 0$, to get new fourth order

¹The name comes from the fundamental role of the linearized KS operator $-\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2$ in the nonlinear SPDE (1.1).

²As in Walsh [43], we treat space-time white noise as a continuous orthogonal martingale measure, and we denote it by \mathscr{W} .

SPDEs with a KPP type nonlinearity³. We then use our explicit-kernel formulation to obtain, among other things, existence, uniqueness, and dimension-dependent Hölder regularity results with sharp spatio-temporal Hölder exponents for versions of the fourth order SPDE (1.1). More specifically, our first result Theorem 1.1 establishes existence, uniqueness, and sharp dimension-dependent Hölder regularity for the zero drift ($b \equiv 0$ or canonical L-KS SPDE) fixed (ε, ϑ) version of (1.1); Theorem 1.2 gives dimension-dependent order parameters limiting results on the competing interaction between the linearized Kuramoto-Sivashinsky (L-KS) operator $-\frac{\varepsilon}{8}(\Delta + 2\vartheta)^2$ and the white noise \mathscr{W} in (1.1) with $b \equiv 0$ and ϑ fixed, and it gives the precise order parameters rate controlling whether the L^p distance between an L-KS SPDE and its corresponding L-KS PDE uniformly vanishes (as the rate goes to zero) or whether there is a finite-time L^2 blowup of L-KS SPDEs (as the rate goes to infinity); and Theorem 1.3 adapts our earlier space-time change of measure results, with widely applicable conditions—from the second order SPDEs case [13, 12, 11] to our fourth order SPDEs setting here—to transfer uniqueness in law and establish the law equivalence between the zero drift ($b \equiv 0$) and the nonlinear nonzero drift versions of (1.1) on $\{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3$ and compact rectangles thereof. This allows us to transfer almost sure properties of solutions—including regularity—between linear and nonlinear L-KS SPDEs in spatial dimensions $d = 1, 2, 3$. An important special case covered by Theorem 1.3 is the aforementioned Swift-Hohenberg SPDE.

We note here that the deterministic Swift-Hohenberg PDE (both real and complex) models numerous pattern formation phenomena in physics, chemistry, and optics (see e.g. [18, 23, 33, 36, 38, 39, 40, 45]). These include the Taylor-Couette flow, the Rayleigh-Bénard convection problem in a horizontal fluid layer in the gravitational field, large-scale flows and spiral core instabilities, and some chemical reactions. Also, in optics, this equation is connected to spatial structures in large aspect lasers and synchronously pumped optical parametric oscillators. The noisy Swift-Hohenberg PDE (or SPDE) treated here in Theorem 1.3 and Corollary 1.1 is at least as interesting and applicable. We also remark that we use our kernel representational approach in separate papers to investigate time asymptotics and other qualitative behavior of a class of fourth order equations with different nonlinearities.

Notation 1.1. We sometimes denote by $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ the SPDE (1.1) on subsets of $\{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3$. Similarly, the zero drift case is denoted by $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$.

Before precisely stating our results, it is instructive to motivate and put together the building blocks—and give the different interesting links—in our approach; and then give our solution formulation and definition.

1.1. The L-KS PDE and L-KS kernel. We start with what we call the (ε, ϑ) linearized Kuramoto-Sivashinsky (L-KS) PDE

$$(1.2) \quad \frac{\partial u}{\partial t} = -\frac{\varepsilon}{8}(\Delta + 2\vartheta)^2 u; \quad t > 0, x \in \mathbb{R}^d, d \in \mathbb{N} = \{1, 2, 3, \dots\},$$

and we observe that it is a fundamental part of and/or intimately connected to a large family of interesting linear, nonlinear, deterministic, and stochastic PDEs. This family includes, but is not limited to, both prominent and new compelling fourth order equations—including pattern formation equations—like

³The corresponding deterministic PDEs are, of course, obtained by simply setting $a \equiv 0$.

- (1) the PDE⁴ $\partial_t u = -\frac{\varepsilon}{8}(\Delta + 2\vartheta)^2 u + b(u)$, which includes the Swift-Hohenberg (SH) PDE (when $b(u)$ is an Allen-Cahn type nonlinearity and $\varepsilon, \vartheta > 0$) as well as many other interesting equations;
- (2) variants/versions of the Cahn-Hilliard PDE $\partial_t u = -\frac{\varepsilon}{8}\Delta^2 u + \Delta b(u)$, where b may be an Allen-Cahn type nonlinearity $b(u) = \sum_{k=0}^{2p-1} c_k u^k$, $p \in \mathbb{N}$, and $c_{2p-1} < 0$, $\varepsilon > 0$;
- (3) variants/versions of the Kuramoto-Sivashinsky (KS) PDE like

$$(1.3) \quad \begin{aligned} \partial_t u &= -\frac{\varepsilon}{8}\Delta^2 u - \alpha_1 \Delta u - \alpha_2 \nabla(u, u), \text{ and} \\ \partial_t u &= -\frac{\varepsilon}{8}(\Delta + 2\vartheta)^2 u - \frac{1}{2}\nabla(u, u), \end{aligned}$$

where $\alpha_1, \alpha_2, \varepsilon, \vartheta > 0$;

and the stochastic versions of all the above PDEs, as well as many more new and intriguing fourth order (S)PDEs. Some of these nonlinear equations mentioned have been studied, and continue to be studied, extensively in the deterministic literature (e.g., [21, 22, 29, 30, 34, 41, 42] and the SH references above) and is catching up on the still growing stochastic side (e.g., [27, 28, 44, 46, 45]), where the effect of the noise on the qualitative behavior of the underlying PDEs is of great interest. When they are studied in the presence of a driving space-time white noise—with only few exceptions like [27] and, recently, our work on higher order stochastic equations [1, 2, 5]—these fourth order equations are invariably restricted to one spatial dimension $d = 1$. On the other hand, in our earlier work [10, 9, 7]; we introduced and connected a large class of processes—in which the time parameter is replaced in different ways by a Brownian motion—to new memory-preserving (*memoryful*) fourth order PDEs and to the linearized KS PDE (1.2) with $\varepsilon = \vartheta = 1$:

$$(1.4) \quad \begin{cases} \frac{\partial u}{\partial t} = -\frac{1}{8}\Delta^2 u - \frac{1}{2}\Delta u - \frac{1}{2}u, & (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

in all spatial dimensions⁵ $d \geq 1$ for suitably regular initial data u_0 . At the heart of our approach in [7] is the kernel $\mathbb{K}_{t;x,y}^{\text{LKS}^d}$ —associated with what in [7] we call the imaginary-Brownian-time-Brownian-angle process (IBTBAP)—defined by

$$(1.5) \quad \begin{cases} K_{\mathbf{i}s;x,y}^{\text{ASP}^d} := \exp(\mathbf{i}s) \frac{e^{-|x-y|^2/2\mathbf{i}s}}{(2\pi\mathbf{i}s)^{d/2}}, \\ \mathbb{K}_{t;x,y}^{\text{LKS}^d} := \int_{-\infty}^0 K_{\mathbf{i}s;x,y}^{\text{ASP}^d} K_{t;s}^{\text{BM}} ds + \int_0^\infty K_{\mathbf{i}s;x,y}^{\text{ASP}^d} K_{t;s}^{\text{BM}} ds \end{cases}$$

where $\mathbf{i} = \sqrt{-1}$ and $K_{t;s}^{\text{BM}} = \frac{e^{-s^2/2t}}{\sqrt{2\pi t}}$. Since $\mathbb{K}_{t;x}^{\text{LKS}^d}$, obtained by setting $y = 0 \in \mathbb{R}^d$ in $\mathbb{K}_{t;x,y}^{\text{LKS}^d}$, is the fundamental solution of the L-KS PDE in (1.4), we also call it the L-KS kernel⁶. Quantum mechanics experts will quickly recognize that, except for

⁴Throughout the article we alternate freely between the notations ∂_x and $\partial/\partial x$ (or d/dx) for partial (or full) derivatives, with respect to any variable x , for aesthetic and typesetting reasons.

⁵This is important to note since one of the major challenges in the study of the nonlinear KS equation is that the existence of solutions in spatial dimensions $d \geq 2$ is unsettled, even in the noiseless deterministic case (see [41]).

⁶See also Section 1.3 and Section 2 below for a simpler form of $\mathbb{K}_{t;x}^{\text{LKS}^d}$ and its connection to (1.4) via Fourier transforms.

the $\exp(\mathbf{i}s)$ angle term, $K_{\mathbf{i}s;x,y}^{\text{ASP}^d}$ in the definition of the L-KS kernel in (1.5) is a d -dimensional version of the free propagator associated with Schrödinger's equation. It is then proved in Theorem 1.1 of [7] that, for⁷ $u_0 \in C_c^{2,\gamma}(\mathbb{R}^d; \mathbb{R})$,

$$(1.6) \quad u(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{LKS}^d} u_0(y) dy$$

solves the linearized Kuramoto-Sivashinsky PDE (1.4) and hence that the kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$ solves the PDE L-KS in (1.4) with initial condition $\delta(x)$.

1.2. Imaginary-Brownian-time-Brownian-angle and Schrödinger links. An important intuitive ingredient in the formulation and proof of Theorem 1.1 of [7], and in arriving at the kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$, was the use of the intimate connection between the Brownian-time processes and their densities in [10, 9] and the imaginary-Brownian-time-Brownian-angle process and its kernel⁸ in [7]. Our IBTBAP process, starting at $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$, was given in [7] by

$$(1.7) \quad \mathbb{A}_{u_0}^{X,B}(t, x) := \begin{cases} u_0(X^x(\mathbf{i}B(t))) \exp(\mathbf{i}B(t)), & B(t) \geq 0; \\ u_0(\mathbf{i}X^{-\mathbf{i}x}(-\mathbf{i}B(t))) \exp(\mathbf{i}B(t)), & B(t) < 0; \end{cases}$$

where the process X^x is an \mathbb{R}^d -valued Brownian motion (BM) starting from $x \in \mathbb{R}^d$, $X^{-\mathbf{i}x}$ is an independent $i\mathbb{R}^d$ -valued BM starting at $-ix$ (so that $iX^{-\mathbf{i}x}$ starts at x), and both are independent of the inner standard \mathbb{R} -valued Brownian motion B starting from 0. The clock of the outer Brownian motions X^x and $X^{-\mathbf{i}x}$ is replaced by an imaginary positive Brownian time; and the angle of $\mathbb{A}_{u_0}^{X,B}(t, x)$ is the Brownian motion B . We think of the imaginary-time processes $\{X^x(is), s \geq 0\}$ and $\{iX^{-\mathbf{i}x}(-is), s \leq 0\}$ as having the same complex Gaussian distribution on \mathbb{R}^d with the corresponding complex distributional density (or Schrödinger propagator)

$$K_{\mathbf{i}s;x,y}^{\text{SP}^d} = \frac{1}{(2\pi is)^{d/2}} e^{-|x-y|^2/2is}.$$

We may then think of u in (1.6) in terms of complex expectation by first conditioning on $B(t) = s$ and then removing the conditioning (by integrating over s) and defining $u(t, x) := \mathbb{E}^{\mathbb{R}}[\mathbb{A}_{u_0}^{X,B}(t, x)]$. Viewed this way, $\mathbb{K}_{t;x}^{\text{LKS}^d}$ is the expectation kernel of the IBTBAP. Since $K_{\mathbf{i}s;x,y}^{\text{ASP}^d} = e^{\mathbf{i}s} K_{\mathbf{i}s;x,y}^{\text{SP}^d}$ is obtained by giving the propagator $K_{\mathbf{i}s;x,y}^{\text{SP}^d}$ an extra angle $s \in \mathbb{R} \setminus \{0\}$, where s is also the real-valued time on the imaginary axis ($\mathbf{i}s$), we call $K_{\mathbf{i}s;x,y}^{\text{ASP}^d}$ the d -dimensional \mathbb{R} -time-angled propagator. The L-KS kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$ in (1.5) is thus the Gaussian average of an \mathbb{R} -time-angled Schrödinger propagator⁹.

⁷The compact support assumption on u_0 here and in Theorem 1.1 below is for convenience only and may be replaced with more relaxed integrability conditions à la those given for the Brownian-time Brownian sheet in [3].

⁸In particular, Theorem 1.2 in [9] was crucial in arriving at our IBTBAP and its kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$. Of course, the L-KS kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$ is *not* a proper probability density in the standard sense. But, it has a nice Fourier transform, as we shall see shortly.

⁹In our fourth order setting we have two notions of time: the standard time t and the Brownian-time and Brownian-angle $B(t)$ (each s in $K_{\mathbf{i}s;x,y}^{\text{ASP}^d}$ represents a possible value for the BM B in our IBTBAP at some time t , $B(t) = s \in \mathbb{R}$).

1.3. The (ε, ϑ) L-KS kernel formulation. In this article, we start by using our L-KS kernel to formulate the notion of a mild kernel solution to the (ε, ϑ) L-KS (S)PDEs in (1.1). We first generalize slightly $\mathbb{K}_{t;x}^{\text{LKS}^d}$ in (1.5) by scaling the time t with a parameter $\varepsilon > 0$ and scaling the angle s in the \mathbb{R} -time-angled propagator $K_{\mathbf{s};x,y}^{\text{ASP}^d}$ by another parameter $\vartheta \in \mathbb{R}$ to obtain the (ε, ϑ) L-KS kernel¹⁰

$$(1.8) \quad \mathbb{K}_{t;x,y}^{(\varepsilon,\vartheta)\text{LKS}^d} = \int_{-\infty}^0 \frac{e^{i\vartheta s} e^{-|x-y|^2/2\mathbf{i}s}}{(2\pi\mathbf{i}s)^{d/2}} K_{\varepsilon t;s}^{\text{BM}} ds + \int_0^\infty \frac{e^{i\vartheta s} e^{-|x-y|^2/2\mathbf{i}s}}{(2\pi\mathbf{i}s)^{d/2}} K_{\varepsilon t;s}^{\text{BM}} ds,$$

which, when setting $y = 0 \in \mathbb{R}^d$, is the fundamental solution $\mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d}$ to the (ε, ϑ) L-KS PDE in equation (1.2)¹¹. As we will see in Section 2, despite the involved expression in (1.8), the kernel $\mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d}$ has a rather nice (and revealing) Fourier transform:

$$(1.9) \quad \hat{\mathbb{K}}_{t;\xi}^{(\varepsilon,\vartheta)\text{LKS}^d} = (2\pi)^{-\frac{d}{2}} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2}; \quad \varepsilon > 0, \quad \vartheta \in \mathbb{R},$$

which, upon inverting yields the simpler form of $\mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d}$

$$(1.10) \quad \begin{aligned} \mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d} &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2} e^{\mathbf{i}\xi \cdot x} d\xi; \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2} \cos(\xi \cdot x) d\xi; \quad \varepsilon > 0, \quad \vartheta \in \mathbb{R}. \end{aligned}$$

The last equality in (1.10) follows since $\int_{\mathbb{R}^d} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2} \sin(\xi \cdot x) d\xi = 0$. Thus, the effect of the Gaussian average of the propagator in (1.8) is to “average out” the imaginary part of the kernel, and $\mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d}$ is real-valued. We now give the new kernel formulation for the class of (ε, ϑ) L-KS (S)PDEs (1.1) which includes, among many other (S)PDEs, the Swift-Hohenberg (S)PDE.

Definition 1.1 (ε, ϑ) L-KS kernel (mild) formulation of (ε, ϑ) L-KS (S)PDEs (1.1). Fix $\varepsilon > 0$ and $\vartheta \in \mathbb{R}$. We call the pair (U, \mathscr{W}) on a usual probability space¹² $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$ a (ε, ϑ) L-KS kernel solution to (1.1) on $\mathbb{R}_+ \times \mathbb{R}^d$ whenever \mathscr{W} is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$; the random field U is progressively measurable, and with $U(0, x) = u_0(x)$; and the pair (U, \mathscr{W}) satisfies the (ε, ϑ) L-KS kernel (mild) formulation:

$$(1.11) \quad \begin{aligned} U(t, x) &= \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{(\varepsilon,\vartheta)\text{LKS}^d} u_0(y) dy \\ &+ \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{t-s;x,y}^{(\varepsilon,\vartheta)\text{LKS}^d} [b(U(s, y)) ds dy + a(U(s, y)) \mathscr{W}(ds \times dy)] \end{aligned}$$

¹⁰Clearly, using the notation of the-just-introduced (ε, ϑ) L-KS kernel, we note that $\mathbb{K}_{t;x}^{\text{LKS}^d} = \mathbb{K}_{t;x}^{(1,1)\text{LKS}^d}$.

¹¹See Section 2 for a Fourier argument. We also briefly note that with $\varepsilon > 0$, we always have the dissipative negative biLaplacian $-\Delta^2$ in (1.2). On the other hand, the case $\vartheta < 0$ leads to a dissipative second order Δ ; whereas $\vartheta > 0$ leads to the non-dissipative second order $-\Delta$, which is the case in L-KS PDEs like the Swift-Hohenberg and Kuramoto-Sivashinsky for example.

¹²We assume throughout the article that filtrations $\{\mathscr{F}_t\}_{t \geq 0}$ satisfies the usual conditions, and we often simply say that U is a kernel solution to (1.1) to mean the same as the definition of a mild solution above.

for $t > 0$ and for every (or almost every) $x \in \mathbb{R}^d$, almost surely \mathbb{P} . Weak and strong—in the probability sense—solutions are defined in the usual way: we call a (ε, ϑ) L-KS kernel solution weak if the white noise \mathscr{W} and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ on which it's defined are freely chosen—along with U —so as to satisfy (1.11); and the solution is strong if \mathscr{W} and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ are fixed and $\{\mathcal{F}_t\}$ is the augmentation of the natural filtration for \mathscr{W} under \mathbb{P} . The solution is continuous if U has continuous paths on $\mathbb{R}_+ \times \mathbb{R}^d$ almost surely \mathbb{P} .

Uniqueness in law holds for (1.1) if the laws¹³ $\mathcal{L}_{\mathbb{P}^{(i)}}^{U^{(i)}}$ of $U^{(i)}$ under $\mathbb{P}^{(i)}$; $i = 1, 2$, are the same whenever $(U^{(i)}, \mathscr{W}^{(i)})$, $(\Omega^i, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathbb{P}^i)$; $i = 1, 2$, are (ε, ϑ) L-KS kernel solutions to (1.1). Pathwise uniqueness holds if $U^{(1)}$ and $U^{(2)}$ are \mathbb{P} -indistinguishable ($\mathbb{P}[U^{(1)} = U^{(2)}] = 1$) whenever $(U^{(i)}, \mathscr{W})$ are (ε, ϑ) L-KS kernel solutions to (1.1) with respect to the same white noise \mathscr{W} and on the same probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$.

A (ε, ϑ) L-KS kernel solution U to the deterministic version of (1.1) is obtained from (1.11) by setting $a \equiv 0$.

Remark 1.1. Although we focus in this article on the SPDE in (1.1) on $\{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3$ and subsets thereof, the utility of our new L-KS kernel formulation goes well beyond just (1.1). We show separately how to adapt it to formulate the class of PDEs discussed in Section 1.1 (Cahn-Hilliard, Kuramoto-Sivashinsky, and many other fourth order equations) and their stochastic versions. We also use this explicit kernel approach in separate articles to analyze the time-asymptotic¹⁴ and other qualitative behaviors of several fourth order L-KS type equations.

1.4. Three main theorems and a Swift-Hohenberg corollary. In this article, we establish three main theorems on versions of the (ε, ϑ) L-KS SPDE (1.1). We now detail and state our main results.

1.4.1. Theorem 1.1: existence, uniqueness, and sharp dimension-dependent Hölder regularity. In our first result; we obtain sharp, dimension-dependent, spatio-temporal Hölder continuity regularity results for the L-KS SPDE (1.1) with $b \equiv 0$ (zero drift or canonical L-KS SPDE):

$$(1.12) \quad \begin{cases} \frac{\partial U}{\partial t} = -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 U + a(U) \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\ U(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

In particular, for any fixed $\varepsilon, T > 0$ and $\vartheta \in \mathbb{R}$, we obtain the existence of a unique real-valued solution U that is $L^p(\Omega)$ -bounded on $[0, T] \times \mathbb{R}^d$ for all $p \geq 2$ and that has Hölder continuous paths in time and space. In time, the Hölder exponent is $\gamma_t \in (0, (4-d)/8)$ and in space it is $\gamma_s \in (0, [(4-d)/2] \wedge 1)$, for spatial dimensions $d = 1, 2, 3$. We first obtained the same striking spatio-temporal Hölder regularity profile in [2] for a different class of memoryful fourth order stochastic integral equations (SIEs) associated with the Brownian-time Brownian motion (BTBM)—see [10, 9, 5] and the discussion in [2]—which we introduced as BTBM SIEs. What is remarkable about this Hölder regularity profile is that, not only

¹³All solutions U in this article have continuous paths ($U \in C(\mathbb{D}; \mathbb{R})$, where $\mathbb{D} \subset \mathbb{R}_+ \times \mathbb{R}^d$). The law $\mathcal{L}_{\mathbb{P}}^U$ of the random field U under \mathbb{P} is the probability measure induced on the Borel σ -field of continuous function by the recipe: $\mathcal{L}_{\mathbb{P}}^U(\Lambda) = \mathbb{P}[U \in \Lambda]$, $\Lambda \in \mathcal{B}(C(\mathbb{D}; \mathbb{R}))$

¹⁴See [6] for a similar approach in studying random attractors for the second order Allen-Cahn case.

random field solutions exist in spatial dimensions $d = 1, 2, 3$ (not just for $d = 1$) in the presence of the rough driving space-time white noise¹⁵, but these random field solutions are spatially twice as smooth as the underlying Brownian sheet¹⁶ in $d = 1, 2$. In the followup article [1], we showed that this third dimensionality limit on random field existence and the above spatial Hölder smoothness are maximal¹⁷ in equations driven by a space-time white noise that are first order in time and high order in space—no matter how high the order is in these equations.

Although our L-KS SPDEs here have the same spatio-temporal Hölder profile as the BTBM SIE of [2], proving it by directly adapting our methods in [2] to the L-KS kernel is demanding. The difficulty lies in the fact that the L-KS kernel in (1.8) is the Gaussian average of the highly oscillatory angled complex propagator; whereas the BTBM probability density

$$(1.13) \quad \mathbb{K}_{t;x,y}^{\text{BTBM}^d} = 2 \int_0^\infty \frac{e^{-|x-y|^2/2s}}{(2\pi s)^{d/2}} \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} ds$$

is a Gaussian average of another non-oscillatory Gaussian density. Also, the L-KS kernel is not a proper probability density function as the BTBM density. Thus, we proceed differently by applying a harmonic analytic step to the L-KS kernel at the outset. This turns out to be a useful first step towards obtaining the required regularity estimates. We then use delicate analysis, including comparing the nonzero ϑ angle case to that of the simpler $\vartheta = 0$ case and adapting the probabilistic-analytic arguments from [2] to our setting here, to prove the estimates needed for the proof of Theorem 1.1.

In the process, we give a harmonic analytic explanation of why the BTBM density—which is associated with the quite different memory-preserving *positive biLaplacian* fourth order PDE

$$(1.14) \quad \begin{cases} \frac{\partial u}{\partial t} = \frac{\Delta u_0}{\sqrt{8\pi t}} + \frac{1}{8} \Delta^2 u; & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\ u(0, x) = u_0(x); & x \in \mathbb{R}^d, \end{cases}$$

¹⁵This is in contrast to second order PDEs driven by space-time white noise whose random field solution exists only in $d = 1$. Also, it is noteworthy that, with very few exceptions (e.g., [1, 2, 5, 27]), space-time white noise driven SPDEs, even higher order ones, are not treated in more than one spatial dimension.

¹⁶Our article [2] gave the first example of space-time white noise driven equations whose solutions are smoother in either time or space than the underlying Brownian sheet corresponding to the driving white noise.

¹⁷Maximal if the spatial dimension is integer.

and its equivalent¹⁸ time-fractional PDE

$$(1.15) \quad \begin{cases} \partial_t^{\frac{1}{2}} u = \frac{1}{\sqrt{8}} \Delta u; & t \in (0, \infty), x \in \mathbb{R}^d, \\ u(0, x) = u_0(x); & x \in \mathbb{R}^d, \end{cases}$$

where $\partial_t^{\frac{1}{2}}$ is the Caputo fractional derivative—has the same regularizing effect as that of the L-KS kernel and its associated PDE (1.4). This harmonic explanation is given in Section 2 below. A different probabilistic heuristic argument was given in [2] as to why the BTBM-SIEs in [5, 2] are cousins of the L-Kuramoto-Sivashinsky SPDEs here. The Fourier transform of the L-KS kernel, and its inverse, are also used in Section 2 to sketch a different proof of the L-KS PDE (1.4) connection to the L-KS kernel, first proved differently in [7].

The regularity—and other qualitative behavior—results carry over to a large class of nonlinear L-KS SPDEs like (1.1) and others intimately connected to the linear KS operator $-\frac{\varepsilon}{8}(\Delta + 2\vartheta)^2$. Some of these are illustrated in Theorem 1.3, which is possible by adapting our earlier change of measure results [13, 12, 11] from the second order to the fourth order settings. Understanding the L-KS PDE (1.4) and SPDE (1.12) is thus very useful in understanding a large class of nonlinear L-Kuramoto-Sivashinsky equations; including the Swift-Hohenberg and its generalization (1.1), variants of the KS, and many more.

Throughout this paper, fix an arbitrary $T > 0$, and let $\mathbb{T} = [0, T]$. We denote by $H^{\gamma_t^-, \gamma_s^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R})$ the space of real-valued locally Hölder functions on $\mathbb{T} \times \mathbb{R}^d$ whose time and space Hölder exponents are in $(0, \gamma_t)$ and $(0, \gamma_s)$, respectively. We now state our first existence, uniqueness, and regularity result¹⁹.

Theorem 1.1 (Existence/uniqueness and sharp Hölder regularity for the canonical (ε, ϑ) L-KS (1.12) in $d = 1, 2, 3$). *Fix $\varepsilon > 0$ and $\vartheta \in \mathbb{R}$. Assume that*

$$(Lip) \quad \begin{cases} (a) & |a(u) - a(v)| \leq C|u - v| \quad u, v \in \mathbb{R}; \\ (b) & a^2(u) \leq C(1 + |u|^2), \quad u \in \mathbb{R}; \\ (c) & u_0 \in C_c^{2, \gamma}(\mathbb{R}^d; \mathbb{R}) \text{ and nonrandom } \forall d \in \{1, 2, 3\}. \end{cases}$$

Then, there exists a strong (ε, ϑ) L-KS kernel solution (U, \mathcal{W}) to the L-KS SPDE (1.12) on $\mathbb{R}_+ \times \mathbb{R}^d$, for $d = 1, 2, 3$, which is $L^p(\Omega)$ -bounded on

¹⁸The connection of BTBM to fourth order PDEs (including (1.14)) was first established in our papers [10, 9]. Also, their connection to time-fractional PDEs was first established implicitly via the BTBM connection to the half derivative generator in [10, Theorem 0.5]. The equivalence between a large class of time-fractional and higher order PDEs with memory, including the equivalence between (1.14) and (1.15) when $u_0 \in C_b^{2, \gamma}$, was established explicitly in [16, Theorem 3.1–Theorem 3.6] (see also [35] and [3]). For a further discussion of interesting aspects of these PDEs and their history see also [10] and the introduction in [1]. In the recent multiparameter-time case the reader is referred to [4, 3]. The BTBM scaling and its nonstandard PDEs connection have now attracted a lot of attention, even outside probability and PDEs, as evidenced by the recent physics and mathematical finance articles [19, 20].

¹⁹We remind the reader again that the compact support condition on u_0 may be replaced with more relaxed integrability conditions like those given for the Brownian-time Brownian sheet in [3].

$\mathbb{T} \times \mathbb{R}^d$ for all $p \geq 2$ ($M_p(t) := \sup_{x \in \mathbb{R}^d} \mathbb{E} |U(t, x)|^p \leq C_T$ for $t \in \mathbb{T}$) and

$$U \in H^{\frac{4-d}{8}, \left(\frac{4-d}{2} \wedge 1\right)^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R}); \text{ for } d = 1, 2, 3, \text{ almost surely.}$$

If (V, \mathcal{W}) is another such solution, with respect to the same white noise \mathcal{W} , then, for any $d \in \{1, 2, 3\}$, U and V are indistinguishable: $\mathbb{P}[U(t, x) = V(t, x); (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d] = 1$.

Remark 1.2. (i) We note here that we can adapt our lattice arguments and K-martingale approach in [2] to prove existence of lattice-limits solutions to our L-KS SPDE with the same L^p and Hölder regularity as those of Theorem 1.1 under the weaker non-Lipschitz conditions²⁰

$$(NLip) \quad \begin{cases} (a) \ a(u) \text{ is continuous in } u \quad u \in \mathbb{R}; \\ (b) \text{ and } (c) \text{ same as in (Lip)}; \end{cases}$$

The details are left to the interested reader.

- (ii) If the Lipschitz condition on a is only local, then the uniqueness assertion of Theorem 1.1 holds for the L-KS SPDE (1.12). However, a local Lipschitz condition is not sufficient to guarantee global existence; and the existence of the unique solution U and its Hölder regularity of Theorem 1.1 hold only up to a random (possibly finite) dimension-dependent blowup time τ_d , for which²¹

$$(1.16) \quad \mathbb{P} \left[\limsup_{t \nearrow \tau_d} \sup_x |U(t, x)| = \infty \right] > 0; \quad d = 1, 2, 3.$$

- (iii) The Hölder exponents confirms our assertion in [5, 2] about the intimate relation between our BTBM SIE there and the L-KS SPDE here. They also tell us that the L-KS SPDE is much smoother than the second order heat SPDE whose solution $U \in H^{\frac{1}{4}, \frac{1}{2}}(\mathbb{T} \times \mathbb{R}; \mathbb{R})$ for only $d = 1$.

1.4.2. Theorem 1.2: *L-KS vs white noise in the limit, the case of vanishing intensities.* Consider the (ε, ϑ) L-KS SPDE (1.12). Fix $\vartheta \in \mathbb{R}$, let $\varepsilon = \varepsilon_1 > 0$ be an order parameter, and attach another order parameter $\varepsilon_2 > 0$ to the white noise term to obtain the L-KS SPDE

$$(1.17) \quad \begin{cases} \frac{\partial U}{\partial t} = -\frac{\varepsilon_1}{8} (\Delta + 2\vartheta)^2 U + \varepsilon_2 a(U) \frac{\partial^{d+1} W}{\partial t \partial x}, & (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\ U(0, x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

By changing the values of ε_1 and ε_2 , we are simply changing the intensities of the smoothing L-KS spatial operator $-\frac{1}{8} (\Delta + 2\vartheta)^2$ and the roughening noise term, respectively. Theorem 1.1 gives us existence, uniqueness, sharp Hölder regularity,

²⁰These lattice arguments have their roots in our second order SPDE works [14, 8].

²¹Our work in a separate article indicates that, when $a(u) = |u|^\alpha$, there is a dimension-dependent critical blowup exponent $\alpha_c^{(d)} > 1$ for $d = 1, 2, 3$, such that

(a) if $\alpha > \alpha_c^{(d)}$, then the LKS SPDE (1.12) with diffusion coefficient $a(u) = |u|^\alpha$ blows up in finite time ($\tau_d < \infty$ in (1.16)); and
(b) if $\alpha < \alpha_c^{(d)}$ then finite time blowup does not occur ($\tau_d = \infty$ in (1.16)).

More on that in the aforementioned upcoming article.

and $L^p(\Omega)$ -boundedness on $[0, T] \times \mathbb{R}^d$ for all $T > 0, p \geq 2$, and hence no blowup, for *any* fixed values of $\varepsilon_1, \varepsilon_2$ on *any* time interval $[0, T]$, under Lipschitz conditions on a . Our second result, Theorem 1.2, is concerned with the interesting questions: what happens in the limit to the solution of (1.17) as $\varepsilon_1, \varepsilon_2 \searrow 0$? and how fast can $\varepsilon_1 \searrow 0$ relative to the speed at which $\varepsilon_2 \searrow 0$ before the noise term dominates the L-KS operator? Theorem 1.2 answers these questions by giving a precise critical ratio, $\varepsilon_2/\varepsilon_1^{d/8}$ for dimensions $d = 1, 2, 3$, which controls the limiting behavior of (1.17) as $\varepsilon_1, \varepsilon_2 \searrow 0$. According to Theorem 1.2, this critical ratio tells us that if we let $\varepsilon_1, \varepsilon_2 \searrow 0$ we should be careful *not* to let $\varepsilon_1 \searrow 0$ so fast that $\varepsilon_2/\varepsilon_1^{d/8} \nearrow \infty$ or the supremum—over space and time—of the solution’s $L^2(\Omega)$ norm will grow to infinity on any time interval $[0, T]$. On the other hand, this critical ratio also says that when we let $\varepsilon_1, \varepsilon_2 \searrow 0$ so that $\varepsilon_1 \searrow 0$ slow enough that $\varepsilon_2/\varepsilon_1^{d/8} \searrow 0$, then the solution to (1.17) converges to its deterministic version ($a \equiv 0$) $\varepsilon_1 \searrow 0$ limit in $L^{2q}(\Omega)$ for all ($q \geq 1$). The same phenomenon happens in the case of the less regular second order heat SPDE (see Remark 1.2 (iii))

$$(1.18) \quad \begin{cases} \frac{\partial U}{\partial t} = -\frac{\varepsilon_1}{2} \Delta U + \varepsilon_2 a(U) \frac{\partial^2 W}{\partial t \partial x}, & (t, x) \in (0, +\infty) \times \mathbb{R}; \\ U(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

with a less forgiving critical ratio of $\varepsilon_2/\varepsilon_1^{1/4}$ and only for $d = 1$ ²²—as we showed in [8, Theorem 1.8]. Comparing the two critical ratios for the L-KS SPDE (1.17) and the heat SPDE (1.18), we see another manifestation of the fact that the L-KS fourth order operator has a substantially stronger regularizing effect on white noise—we can afford to turn it off much faster—than does the usual Laplacian in second order noisy heat PDEs. For the critical ratio $\varepsilon_2/\varepsilon_1^{1/8}$ to grow to ∞ as $\varepsilon_1, \varepsilon_2 \searrow 0$ in the one-dimensional L-KS case (1.17), we have to “turn off” the smoothing L-KS operator (let $\varepsilon_1 \searrow 0$) at the fast rate of $\varepsilon_1 = \varepsilon_2^8 z(\varepsilon_2)$ for some function $z > 0$ satisfying $\lim_{\varepsilon_2 \searrow 0} z(\varepsilon_2) = 0$ (e.g., $z(\varepsilon_2) = \varepsilon_2^\delta$ for $\delta > 0$); whereas, in the second order heat SPDE case (1.18), the critical ratio there, $\varepsilon_2/\varepsilon_1^{1/4} \nearrow \infty$ as $\varepsilon_1, \varepsilon_2 \searrow 0$ even if we turn off the smoothing Laplacian at the much slower rate of $\varepsilon_1 = \varepsilon_2^4 z(\varepsilon_2)$. Also, for the critical ratio $\varepsilon_2/\varepsilon_1^{1/8}$ to converge to 0 as $\varepsilon_1, \varepsilon_2 \searrow 0$ in the one-dimensional L-KS case (1.17), we can turn off the L-KS operator at a rate as high as $\varepsilon_1 = \varepsilon_2^8 z^{-1}(\varepsilon_2)$; however, in the second order heat SPDE case, we have to turn off the Laplacian at a much slower rate of no more than $\varepsilon_1 = \varepsilon_2^4 z^{-1}(\varepsilon_2)$ to cause the critical ratio there, $\varepsilon_2/\varepsilon_1^{1/4}$ to converge to 0. In $d = 2, 3$, while the heat SPDE (1.18) does not have random field solutions, the L-KS SPDE (1.17) has Hölder continuous solutions (Theorem 1.1) and a critical ratio $\varepsilon_2/\varepsilon_1^{d/8}$. In particular, in $d = 2$, the critical ratio for the L-KS SPDE is $\varepsilon_2/\varepsilon_1^{1/4}$, the same as the one for the heat SPDE in $d = 1$. This is consistent with the fact that the effective Hölder exponent²³ of the L-KS SPDE in $d = 2$ is the same as that for the heat SPDE in $d = 1$, $(1/4)^-$. In $d = 3$, the critical ratio for the L-KS SPDE is $\varepsilon_2/\varepsilon_1^{3/8}$. Theorem 1.2 rigorizes the above heuristic for the L-KS SPDE (1.17) and gives a precise meaning of “too fast” or “too slow” for the speed of the limit $\varepsilon_1 \searrow 0$, relative to the speed at which the intensity of the noise term ε_2 is vanishing, by

²²As we remarked before, unlike our L-KS SPDEs here, random field solutions to these second order SPDEs are restricted to one dimensional space.

²³The minimum of the temporal and spatial Hölder exponents.

giving the precise critical ratio of ε_2 to ε_1 that determines the limiting behavior as $\varepsilon_1, \varepsilon_2 \searrow 0$.

Theorem 1.2 (L-KS vs white noise in (1.17): the critical order parameter ratio in $d = 1, 2, 3$). *Fix $\vartheta \in \mathbb{R}$. Assume that the conditions in (Lip) are in force and that $(U_{\varepsilon_1, \varepsilon_2}, \mathcal{W})$ is the unique strong solution to the L-KS SPDE (1.17).*

- (i) *(Uniformly vanishing L^{2q} distance between SPDE and PDE as $\varepsilon_2/\varepsilon_1^{d/8} \searrow 0$) Suppose that u_{ε_1} is the solution to the deterministic L-KS PDE obtained from (1.17) by setting $a \equiv 0$, then*

$$\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} |U_{\varepsilon_1, \varepsilon_2}(t, x) - u_{\varepsilon_1}(t, x)|^{2q} \longrightarrow 0; \forall q \geq 1, T > 0$$

as $\varepsilon_1, \varepsilon_2 \searrow 0$, and $\varepsilon_2/\varepsilon_1^{d/8} \searrow 0$ for $d = 1, 2, 3$.

- (ii) *(Supremum L^2 growth to infinity as $\varepsilon_2/\varepsilon_1^{d/8} \nearrow \infty$) Suppose there are constants $K_l, K_u > 0$ such that $K_l \leq a(v) \leq K_u$ for all $v \in \mathbb{R}$; then,*

$$\sup_{0 \leq s \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} |U_{\varepsilon_1, \varepsilon_2}(s, x)|^2 \nearrow \infty, \forall T > 0.$$

as $\varepsilon_1, \varepsilon_2 \searrow 0$ such that the ratio $\varepsilon_2/\varepsilon_1^{d/8} \nearrow \infty$ for $d = 1, 2, 3$.

Both Theorem 1.2 above and [8, Theorem 1.8] capture the same common phenomenon²⁴ for (1.17) and (1.18), respectively, associated with the vanishing of ε_1 and ε_2 . Each theorem gives a different critical ratio that is at the edge of the two extreme behaviors in Theorem 1.2 (i) and (ii) and in [8, Theorem 1.8].

1.4.3. Theorem 1.3: from canonical L-KS SPDEs to nonlinear L-KS SPDEs via change of measure. At their core, our space-time change of measure theorems in [13, 12, 11] are “noise” results that are independent of both the type and order of the SPDE under consideration. This makes them conveniently adaptable to different SPDEs settings. We use this fact to adapt our earlier change of measure results, from the second order equations in [13, 12, 11] to the fourth order equations of this article, to transfer results and properties from the zero drift L-KS SPDE (1.12) (linear PDE part) to the nonzero-drift case (1.1) (nonlinear PDE part). In addition, we use the same almost sure L^2 condition on the drift/diffusion ratio as in our work [12, 11] to transfer uniqueness in law and establish law equivalence between solutions to (1.12) and (1.1). As observed in [12], this is a much weaker condition than the traditional Novikov condition for change of measure; and this allows us to transfer results and properties from the canonical L-KS SPDEs (1.12) to many nonlinear L-KS SPDEs (1.1), including the Swift-Hohenberg SPDE, driven by space-time white noise on subsets of $\mathbb{R}_+ \times \mathbb{R}^d$, $d = 1, 2, 3$.

Now, we turn to the setting of our final main result of this paper. Recall that we denote the zero-drift L-KS SPDE (1.12) by $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ while the SPDE (1.1) is denoted by $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$. We fix $T, L_1, L_2, L_3 > 0$, let $\mathbb{T} = [0, T]$, and we consider both equations $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ and $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ on the time-space domain $\mathbb{T} \times \mathbb{S}$, where either $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \prod_{i=1}^d [0, L_i]$, $d = 1, 2, 3$. In the case $\mathbb{S} =$

²⁴This phenomenon is also shared with the time-fractional/high order stochastic integral equations of [2, 1] with the same (in [2]) or more forgiving (in [1]) critical ratios as the L-KS, depending on the β value.

$\prod_{i=1}^d [0, L_i]$ the equations are supplemented with suitable boundary conditions²⁵, the nature of which is irrelevant to our change of measure results. Also irrelevant to our results below is whether solutions are defined as mild kernel solutions like in Definition 1.1—with appropriate modifications to $\mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d}$ to account for the boundary conditions²⁶ in the case $\mathbb{S} = \prod_{i=1}^d [0, L_i]$ —or whether solutions are defined as weak; i.e., given in the test functions formulation (TFF). For concreteness, and to also give the independently useful TFF for $e_{\text{LKS}}^{(\varepsilon,\vartheta)}(a, b, u_0)$, we take the TFF as our definition of solutions to $e_{\text{LKS}}^{(\varepsilon,\vartheta)}(a, b, u_0)$ for Theorem 1.3 and we now proceed to define it (see Remark 1.3 below about the equivalence of the two formulations). Let the Dirichlet test functions space be given by²⁷

$$(1.19) \quad \Phi_{c,Dir}^\infty(\mathbb{S}; \mathbb{R}) := \begin{cases} \{\varphi \in C^\infty(\mathbb{R}^d; \mathbb{R}); \varphi = \Delta\varphi = 0 \text{ on } \partial\mathbb{S}\}; & \mathbb{S} = \prod_{i=1}^d [0, L_i], \\ C_c^\infty(\mathbb{S}; \mathbb{R}); & \mathbb{S} = \mathbb{R}^d, \end{cases}$$

where $d = 1, 2, 3$.

Definition 1.2 (Test function solutions to $e_{\text{LKS}}^{(\varepsilon,\vartheta)}(a, b, u_0)$). We say that the pair (U, \mathscr{W}) defined on the usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ is a test function solution to $e_{\text{LKS}}^{(\varepsilon,\vartheta)}(a, b, u_0)$ on $\mathbb{R}_+ \times \mathbb{S}$ if \mathscr{W} is a space-time white noise on $\mathbb{R}_+ \times \mathbb{S}$; the random field U is progressively measurable, and with $U(0, x) = u_0(x)$; and the pair (U, \mathscr{W}) satisfies the test function formulation:

$$(1.20) \quad \begin{aligned} (U(t) - u_0, \varphi) &= \int_0^t \left[- \left(U(s), \frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 \varphi \right) + (b(U(s)), \varphi) \right] ds \\ &+ \int_{\mathbb{S}} \int_0^t a(U(s, y)) \varphi(y) \mathscr{W}(ds \times dy); \quad \forall \varphi \in \Phi_{c,Dir}^\infty(\mathbb{S}; \mathbb{R}), t > 0, \text{ a.s. } \mathbb{P}, \end{aligned}$$

where (\cdot, \cdot) denotes the usual inner product on $L^2(\mathbb{S}; \mathbb{R})$. The test function solution is continuous if U has continuous paths on $\mathbb{R}_+ \times \mathbb{S}$. Weak and strong—in the probability sense—solutions and uniqueness in law and pathwise uniqueness are defined in the usual way as in Definition 1.1.

Remark 1.3. We often simply say that U is a test function solution to $e_{\text{LKS}}^{(\varepsilon,\vartheta)}(a, b, u_0)$ (weakly or strongly) to mean the same thing as above. As in Walsh’s treatment of second order SPDEs (the top of p. 314 in [43] and the discussion before it), it is straightforward to show the equivalence of the two formulations: kernel formulation in (1.11) (with spatial set $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \prod_{i=1}^d [0, L_i]$) and test function formulation in (1.20) under local boundedness assumptions on a and b .

²⁵E.g., boundary conditions of Neumann type $\partial U / \partial n = \partial \Delta U / \partial n = 0$ or Dirichlet type conditions $U = \Delta U = 0$ on $\partial\mathbb{S}$ and $d = 1, 2, 3$.

²⁶E.g., in the Neumann (Dirichlet) case, the propagator $e^{-|x-y|^2/2\mathbf{1}s} / (2\pi\mathbf{1}s)^{d/2}$ in the definition of the (ε, ϑ) L-KS kernel $\mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d}$ (1.8) is replaced with the propagator with reflection (absorption) at $\partial\mathbb{S}$, respectively.

²⁷Of course, the Dirichlet choice, which is assumed throughout the article whenever $\mathbb{S} = \prod_{i=1}^d [0, L_i]$, is without loss of generality and for concreteness only. The Neumann (and other) boundary conditions are just as easily handled.

We use λ to denote the Lebesgue measure on $\mathcal{B}(\mathbb{T} \times \mathbb{R}^d)$. Also, for any function $u : \mathbb{T} \times \mathbb{S}$, we use the following notation for the drift/diffusion ratio function:

$$(1.21) \quad R_u(t, x) := \frac{b(u(t, x))}{a(u(t, x))}; (t, x) \in \mathbb{T} \times \mathbb{S}.$$

Theorem 1.3 (From canonical L-KS to nonlinear L-KS SPDEs on subsets of $\{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3$ via change of measure). *Assume that either $\mathbb{S} = \mathbb{R}^d$ or $\mathbb{S} = \prod_{i=1}^d [0, L_i]$, $d = 1, 2, 3$. Suppose that the ratios R_U and R_V are in $L^2(\mathbb{T} \times \mathbb{S}, \lambda)$, almost surely, whenever the continuous random fields U and V solve (weakly or strongly) $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ and $e_{LKS}^{(\varepsilon, \vartheta)}(a, b, u_0)$, respectively, on $\mathbb{T} \times \mathbb{S}$. Then,*

- (i) *uniqueness in law holds for $e_{LKS}^{(\varepsilon, \vartheta)}(a, b, u_0)$ iff uniqueness in law holds for $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$; and*
- (ii) *if uniqueness in law holds for $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$, U is a solution to $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$, and V is a solution to $e_{LKS}^{(\varepsilon, \vartheta)}(a, b, u_0)$ on $\mathbb{T} \times \mathbb{S}$; then the laws of U and V on $\mathcal{B}(C(\mathbb{T} \times \mathbb{S}; \mathbb{R}))$ are equivalent (mutually absolutely continuous).*

In particular, let $\mathbb{S} = \prod_{i=1}^d [0, L_i]$, $d = 1, 2, 3$; and assume that $a(u) = \kappa \in \mathbb{R} \setminus \{0\}$, $b(u) = \sum_{k=0}^N c_k u^k$ for $c_k \in \mathbb{R}$, $k = 0, \dots, N$, and $N \geq 0$, and $u_0 \in C_c^{2,\gamma}(\mathbb{S}; \mathbb{R})$ and nonrandom. Then, the conclusions in (i) and (ii) above hold, uniqueness in law holds for $e_{LKS}^{(\varepsilon, \vartheta)}(a, b, u_0)$, and if U and V are continuous solutions to $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ and $e_{LKS}^{(\varepsilon, \vartheta)}(a, b, u_0)$, respectively, on $\mathbb{T} \times \mathbb{S}$

$$\begin{aligned} V &\in H^{\frac{4-d}{8}-, \left(\frac{4-d}{2} \wedge 1\right)-}(\mathbb{T} \times \mathbb{S}; \mathbb{R}) \text{ for } d = 1, 2, 3 \text{ a.s.} \\ \iff U &\in H^{\frac{4-d}{8}-, \left(\frac{4-d}{2} \wedge 1\right)-}(\mathbb{T} \times \mathbb{S}; \mathbb{R}) \text{ for } d = 1, 2, 3 \text{ a.s.} \end{aligned}$$

The change of measure equivalence of Theorem 1.3—between the canonical L-KS SPDE $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ and a large class of nonlinear L-KS SPDEs $e_{LKS}^{(\varepsilon, \vartheta)}(a, b, u_0)$ —immediately leads to uniqueness in law for the important special case ($N = 2p - 1$, $p \in \mathbb{N}$ and $c_{2p-1} < 0$) of the Swift-Hohenberg SPDE and its generalization:

$$(1.22) \quad e_{LKS}^{(\varepsilon, \vartheta)}(a, b, u_0), \text{ with } b(u) = \sum_{k=0}^{2p-1} c_k u^k \text{ and with } p \in \mathbb{N} \text{ and } c_{2p-1} < 0.$$

It also says that the Swift-Hohenberg SPDE shares the same dimension-dependent local Hölder regularity with the canonical L-KS SPDE $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$, as in Theorem 1.3. Even more, the law of the solution of the canonical L-KS SPDE is equivalent to that of the solution to the SH SPDE on $\mathcal{B}(C(\mathbb{T} \times \mathbb{S}; \mathbb{R}))$, where $\mathbb{S} = \prod_{i=1}^d [0, L_i]$, $d = 1, 2, 3$.

Corollary 1.1 (Swift-Hohenberg uniqueness and law equivalence to the canonical L-KS SPDE). *Let $\mathbb{S} = \prod_{i=1}^d [0, L_i]$, $d = 1, 2, 3$, and assume that $a(u) = \kappa \in \mathbb{R} \setminus \{0\}$ and $u_0 \in C_c^{2,\gamma}(\mathbb{S}; \mathbb{R})$ and nonrandom. The generalized Swift-Hohenberg*

SPDE (1.22) admits uniqueness in law and is law equivalent to $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ on $\mathcal{B}(C(\mathbb{T} \times \mathbb{S}; \mathbb{R}))$; consequently, it has the same Hölder regularity as the canonical L-KS SPDE $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ on $\mathbb{T} \times \mathbb{S}$.

Remark 1.4. We note that the conclusions of the last part of Theorem 1.3 hold also in the multiplicative noise case $a(u) = \kappa u$ and $b(u) = \sum_{k=1}^N c_k u^k$, where $\kappa \in \mathbb{R} \setminus \{0\}$ and $c_i \neq 0$ for at least one $i \in \mathbb{N}$ (which covers the standard Allen-Cahn nonlinearity $u(1 - u^2)$ encountered in the SH equation). We note here that all is needed is (1) uniqueness in law for $e_{LKS}^{(\varepsilon, \vartheta)}(a, 0, u_0)$, which holds since the stronger pathwise uniqueness holds because $a(u) = \kappa u$ satisfies (Lip) in Theorem 1.1 and (2) the ratios R_U are R_V are clearly in $L^2(\mathbb{T} \times \mathbb{S}, \lambda)$ by the continuity assumption on U and V and the nonzero assumption on the constants κ and the c_i 's²⁸.

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2. A HARMONIC CONNECTION BETWEEN THE L-KS AND THE BTBM KERNELS

2.1. Fourier transforms and (ε, ϑ) L-KS PDEs links. We start by obtaining the spatial Fourier transforms²⁹ for the Brownian-time Brownian motion (BTBM) and the (ε, ϑ) L-KS kernels. This reveals and captures both similarities and differences between both kernels and the PDEs corresponding to them.

Lemma 2.1 (Spatial Fourier transforms of the BTBM and the (ε, ϑ) L-KS kernels).

Let $\mathbb{K}_{t;x}^{\text{BTBM}^d}$ and $\mathbb{K}_{t;x}^{(\varepsilon, \vartheta)\text{LKS}^d}$ be the BTBM and (ε, ϑ) L-KS kernels, respectively.

(i) The spatial Fourier transform of $\mathbb{K}_{t;x}^{\text{BTBM}^d}$ is given by

$$(2.1) \quad \hat{\mathbb{K}}_{t;\xi}^{\text{BTBM}^d} = (2\pi)^{-\frac{d}{2}} e^{\frac{t}{8}|\xi|^4} \left[\frac{2}{\sqrt{\pi}} \int_{\frac{\sqrt{2t}|\xi|^2}{4}}^{\infty} e^{-\tau^2} d\tau \right].$$

(ii) The spatial Fourier transform of $\mathbb{K}_{t;x}^{(\varepsilon, \vartheta)\text{LKS}^d}$ is given by

$$(2.2) \quad \hat{\mathbb{K}}_{t;\xi}^{(\varepsilon, \vartheta)\text{LKS}^d} = (2\pi)^{-\frac{d}{2}} e^{-\frac{\varepsilon t}{8}(-2\vartheta + |\xi|^2)^2}; \quad \varepsilon > 0, \quad \vartheta \in \mathbb{R}.$$

Proof. Starting with the BTBM kernel Fourier transform, we have

$$(2.3) \quad \begin{aligned} \hat{\mathbb{K}}_{t;\xi}^{\text{BTBM}^d} &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \left[2 \int_0^\infty K_{s;x}^{\text{BM}^d} K_{t;s}^{\text{BM}} ds \right] e^{-i\xi \cdot x} dx \\ &= (2\pi)^{-\frac{d}{2}} 2 \int_0^\infty \frac{e^{-s^2/2t}}{\sqrt{2\pi t}} e^{-\frac{\varepsilon}{2}|\xi|^2 s} ds \\ &= (2\pi)^{-\frac{d}{2}} \left[\frac{2e^{\frac{t}{8}|\xi|^4}}{\sqrt{\pi}} \int_{\frac{\sqrt{2t}|\xi|^2}{4}}^{\infty} e^{-\tau^2} d\tau \right], \end{aligned}$$

²⁸Of course we take $R_U|_{U=0} := \lim_{U \rightarrow 0} R_U = \lim_{V \rightarrow 0} R_V := R_V|_{V=0} = c_1/\kappa$ or 0.

²⁹We use the symmetric definition of the Fourier transform. From a Physics point of view, the Fourier transform is taken over position to get energy.

proving part (i). The Fourier transform of the (ε, ϑ) L-KS kernel is now given by

$$\begin{aligned}
 \hat{\mathbb{K}}_{t;\xi}^{(\varepsilon,\vartheta)\text{LKS}^d} &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R} \setminus \{0\}} \frac{e^{i\vartheta s} e^{-|x|^2/2\mathbf{i}s}}{(2\pi\mathbf{i}s)^{d/2}} K_{\varepsilon t;s}^{\text{BM}} ds \right] e^{-\mathbf{i}\xi \cdot x} dx \\
 &= (2\pi)^{-\frac{d}{2}} \int_{-\infty}^0 e^{-\frac{\mathbf{i}s}{2}(-2\vartheta+|\xi|^2)} K_{\varepsilon t;s}^{\text{BM}} ds + \int_0^\infty e^{-\frac{\mathbf{i}s}{2}(-2\vartheta+|\xi|^2)} K_{\varepsilon t;s}^{\text{BM}} ds \\
 &= (2\pi)^{-\frac{d}{2}} \int_0^\infty \frac{e^{-s^2/2\varepsilon t}}{\sqrt{2\pi\varepsilon t}} \left[e^{-\frac{\mathbf{i}s}{2}(-2\vartheta+|\xi|^2)} + e^{\frac{\mathbf{i}s}{2}(-2\vartheta+|\xi|^2)} \right] ds \\
 &= (2\pi)^{-\frac{d}{2}} e^{-\frac{\varepsilon t}{8}(-2\vartheta+|\xi|^2)^2},
 \end{aligned} \tag{2.4}$$

completing the proof of Lemma 2.1. \square

Remark 2.1. The extra factor $\frac{2}{\sqrt{\pi}} \int_{\frac{\sqrt{2t}|\xi|^2}{4}}^\infty e^{-\tau^2} d\tau$ in the BTBM transform (2.1) capture the memoryful property of the PDE (1.14) (the inclusion of u_0) and the plus sign of the term $t|\xi|^4/8$ corresponds to that of the biLaplacian in (1.14).

Inverting the Fourier transform in Lemma 2.1 we immediately get the more-convenient form for $\mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d}$ in (1.10), which can easily be verified to be a solution to the (ε, ϑ) L-KS PDE in (1.2) with Dirac initial condition $\delta(x)$. In particular, the special case $(\varepsilon, \vartheta) = (1, 1)$ confirms that our L-KS kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$ in (1.5) is the fundamental solution of the L-KS PDE in (1.4). Let u be given by

$$u(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d} u_0(y) dy \tag{2.5}$$

and assume u_0 satisfies the regularity conditions in (Lip) (c). The dominated convergence theorem plus a bit of analysis³⁰ then give us that

$$\begin{aligned}
 \partial_t u(t, x) &= \int_{\mathbb{R}^d} \partial_t \mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d} u_0(y) dy = \int_{\mathbb{R}^d} -\frac{\varepsilon}{8} (\Delta_x + 2\vartheta)^2 \mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d} u_0(y) dy \\
 &= -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 \int_{\mathbb{R}^d} \mathbb{K}_{t;x}^{(\varepsilon,\vartheta)\text{LKS}^d} u_0(y) dy = -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 u(t, x)
 \end{aligned} \tag{2.6}$$

and $u(0, x) = u_0(x)$. Thus, we obtain the following theorem summarizing the PDEs connections.

Theorem 2.1. *The (ε, ϑ) L-KS kernel solves the initial value (ε, ϑ) L-KS PDE*

$$\begin{aligned}
 \frac{\partial u}{\partial t} &= -\frac{\varepsilon}{8} (\Delta + 2\vartheta)^2 u, t > 0, x \in \mathbb{R}^d; \\
 u(0, x) &= \delta(x), x \in \mathbb{R}^d.
 \end{aligned} \tag{2.7}$$

Moreover, if u is given by (2.5), and u_0 satisfies the condition in (Lip) (c), then u solves the (ε, ϑ) L-KS PDE in (2.7) with $u(0, x) = u_0(x)$.

Setting $\varepsilon = \vartheta = 1$ in (2.6) in the argument leading to Theorem 2.1, gives us an alternative proof of our Theorem 1.1 of [7] connecting the linearized KS PDE (1.4)

³⁰See for example Lemma 2.1 in [7]. We leave the very similar details to the interested reader.

to the L-KS kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$. On the other hand, setting $\varepsilon = 1$ and $\vartheta = 0$ in (2.6); we get that the simpler kernel

$$(2.8) \quad \mathbb{K}_{t;x}^{\text{SFO}^d} := \mathbb{K}_{t;x}^{(1,0)\text{LKS}^d} = \int_{-\infty}^0 \frac{e^{-|x-y|^2/2is}}{(2\pi is)^{d/2}} K_{t;0,s}^{\text{BM}} ds + \int_0^\infty \frac{e^{-|x-y|^2/2is}}{(2\pi is)^{d/2}} K_{t;0,s}^{\text{BM}} ds,$$

obtained by removing the angle e^{is} from the L-KS kernel $\mathbb{K}_{t;x}^{\text{LKS}^d}$ in (1.5), is the fundamental solution of the simpler fourth order PDE

$$(2.9) \quad \begin{cases} \frac{\partial u}{\partial t} = -\frac{1}{8}\Delta^2 u, & (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\ u(0, x) = \delta(x), & x \in \mathbb{R}^d \end{cases}$$

as was shown for the case $d = 1$ in Hochberg and Orsinger [32] (see also the different approach in Funaki [31], also for $d = 1$). Clearly, the Fourier transforms $\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}$ and $\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}$ of $\mathbb{K}_{t;x}^{\text{LKS}^d}$ and $\mathbb{K}_{t;x}^{\text{SFO}^d}$, and their inverses are now given as an immediate corollary to Lemma 2.1. Taking $(\varepsilon, \vartheta) = (1, 1)$ and $(\varepsilon, \vartheta) = (1, 0)$, respectively in Lemma 2.1 (ii) and using a dominated convergence argument, we get

Corollary 2.1.

$$(2.10) \quad \begin{aligned} \hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d} &= \frac{e^{-\frac{t}{8}(-2+|\xi|^2)^2}}{(2\pi)^{\frac{d}{2}}}, \quad \mathbb{K}_{t;x}^{\text{LKS}^d} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{t}{8}(-2+|\xi|^2)^2} e^{i\xi \cdot x} d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{t}{8}(-2+|\xi|^2)^2} \cos(\xi \cdot x) d\xi; \\ \hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d} &= \frac{e^{-\frac{t}{8}|\xi|^4}}{(2\pi)^{\frac{d}{2}}}, \quad \mathbb{K}_{t;x}^{\text{SFO}^d} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{t}{8}|\xi|^4} e^{i\xi \cdot x} d\xi \\ &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{t}{8}|\xi|^4} \cos(\xi \cdot x) d\xi. \end{aligned}$$

2.2. A revealing kernels L^2 energy. To understand why the L-KS and the BTBM kernels $\mathbb{K}_{t;x}^{\text{LKS}^d}$ and $\mathbb{K}_{t;x}^{\text{BTBM}^d}$ have very similar regularizing effects on the L-KS SPDE (1.12) above (with $(\varepsilon, \vartheta) = (1, 1)$) and the BTBM SIE introduced in [2] (and obtained from (1.11) by replacing $\mathbb{K}_{t;x}^{(\varepsilon, \vartheta)\text{LKS}^d}$ with $\mathbb{K}_{t;x}^{\text{BTBM}^d}$ and setting $b \equiv 0$), we first observe that the regularity of the L-KS PDE (1.4) is dictated by the bi-Laplacian term and that the family

$$\left\{ \mathbb{K}_{t;x}^{(\varepsilon, \vartheta)\text{LKS}^d} \right\}_{\varepsilon > 0, \vartheta \in \mathbb{R}}$$

of all (ε, ϑ) L-KS kernels in (1.8) and (1.10)—including $\mathbb{K}_{t;x}^{\text{LKS}^d}$ and $\mathbb{K}_{t;x}^{\text{SFO}^d}$ —share the same regularizing effect on the L-KS SPDE (1.12).

As we will see shortly, the L^2 quantity

$$(2.11) \quad \int_{\mathbb{R}^d} \left| \mathbb{K}_{t;x}^{\text{SFO}^d} \right|^2 dx = \int_{\mathbb{R}^d} \left| \hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d} \right|^2 d\xi = \int_{\mathbb{R}^d} \frac{e^{-\frac{t}{4}|\xi|^4}}{(2\pi)^d} d\xi = C_d t^{-d/4}; d = 1, 2, 3,$$

for the $(\varepsilon, \vartheta) = (1, 0)$ L-KS kernel $\mathbb{K}_{t;x}^{\text{SFO}^d}$ —where we used the Parseval-Plancherel theorem and where C_d is a dimension dependent constant³¹—is key to understanding the regularity of our L-KS SPDE (1.12). By the above discussion (see

³¹ $C_1 = 1/2\Gamma(\frac{3}{4})$, $C_2 = 1/4\sqrt{\pi}$, and $C_3 = \Gamma(\frac{3}{4})/\pi^2\sqrt{8}$.

also Lemma 3.1 below), it is clear that $\int_{\mathbb{R}^d} |\mathbb{K}_{t;x}^{\text{LKS}^d}|^2 dx$ is of the same order³². On the other hand as was shown in Lemma 2.2 in [2], there is a dimension dependent constant c_d such that

$$(2.12) \quad \int_{\mathbb{R}^d} \left[\mathbb{K}_{t;x}^{\text{BTBM}^d} \right]^2 dx = c_d t^{-d/4}; \quad t > 0, \quad d = 1, 2, 3.$$

Equations (2.11) and (2.12) are the fundamental analytic reason why the regularity for our L-KS SPDE in our first result Theorem 1.1 above is the same as that of the BTBM SIE in Theorem 1.1 of [2], albeit here we have real solutions to a negative bi-Laplacian equation and the BTBM SIE in [2] has real solutions to a positive bi-Laplacian equation with memory (see [2]).

3. PROOF OF THEOREM 1.1

Since both $\varepsilon > 0$ and $\vartheta \in \mathbb{R}$ are fixed in Theorem 1.1, and since all the main conclusions are unaffected by the specific values of $\varepsilon > 0$ and $\vartheta \in \mathbb{R}$; we will simplify our notation and exposition by assuming throughout this section (and its subsections)—without loss of generality—that either $(\varepsilon, \vartheta) = (1, 1)$ (capturing the general biLaplacian, Laplacian, and zero order term case) or $(\varepsilon, \vartheta) = (1, 0)$ (the biLaplacian term, without the lower order terms, case)³³.

3.1. Key regularity estimates for the L-KS kernel. Here, we prove several L^2 estimates³⁴ on the L-KS kernel and its temporal and spatial differences that are key in proving our regularity results in Theorem 1.1. Again, these fundamental estimates for the L-KS kernel are very similar to those for the BTBM density in the corresponding estimates in [2], but the proofs proceed differently due to the oscillatory nature of the modified propagator part of the L-KS kernel.

Lemma 3.1 (Kernel's L^2). *Fix any arbitrary $T > 0$. There are constants $C_l^{(d)}$ and $C_u^{(d)}$ depending only on the spatial dimension d and T and a constant C_d depending only on d such that*

$$(3.1) \quad \begin{aligned} & \int_{\mathbb{R}^d} \left| \mathbb{K}_{t;x}^{\text{SFO}^d} \right|^2 dx = C_d t^{-d/4}; \text{ and} \\ & C_l^{(d)} t^{-\frac{d}{4}} \leq \int_{\mathbb{R}^d} \left| \mathbb{K}_{s;x}^{\text{LKS}^d} \right|^2 dx \leq C_u^{(d)} t^{-\frac{d}{4}}; \end{aligned}$$

³²In fact, in $d = 2$

$$\int_{\mathbb{R}^d} \left| \hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d} \right|^2 d\xi = \left[1 + \psi(\sqrt{t}) \right] \int_{\mathbb{R}^d} \left| \hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d} \right|^2 d\xi, \quad t > 0$$

where $\psi(u) := (2/\sqrt{\pi}) \int_0^u e^{-r^2} dr$. See also Lemma 3.1 below.

³³It should be clear that our methods extend with only minor notational changes to any fixed values for $\varepsilon > 0$ and $\vartheta \in \mathbb{R}$. The case $(\varepsilon, \vartheta) = (1, 0)$ is the simplest representative case, and we include it explicitly in this subsection since it is useful in Lemma 3.1 to obtain the fundamental L^2 estimates for the more interesting $(\varepsilon, \vartheta) = (1, 1)$ case.

³⁴Lemma 3.1 is stated only for $0 < t \leq T$, since we only need it for intervals $[0, T]$. In fact, for $d = 2$, we show that the estimates hold, with the same constants C_l^2 and C_u^2 , for all $t > 0$.

for $0 < t \leq T$, $d \in \{1, 2, 3\}$ and hence

$$(3.2) \quad \begin{aligned} \int_0^t \int_{\mathbb{R}^d} |\mathbb{K}_{s;x}^{\text{SFO}^d}|^2 dx ds &= C_d t^{\frac{4-d}{4}}; \text{ and} \\ C_l^{(d)} t^{\frac{4-d}{4}} &\leq \int_0^t \int_{\mathbb{R}^d} |\mathbb{K}_{s;x}^{\text{LKS}^d}|^2 dx ds \leq C_u^{(d)} t^{\frac{4-d}{4}}; \end{aligned}$$

for $0 < t \leq T$, $d \in \{1, 2, 3\}$.

Proof. The equalities in (3.1) and in (3.2) follow immediately from (2.11). Using the Parseval-Plancherel theorem, we have

$$(3.3) \quad \int_{\mathbb{R}^d} |\mathbb{K}_{s;x}^{\text{LKS}^d}|^2 dx = \int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{s;\xi}^{\text{LKS}^d}|^2 d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{s}{4}(-2+|\xi|^2)^2} d\xi,$$

for every $s > 0$. Let $d = 2$ and $\psi(u) := (2/\sqrt{\pi}) \int_0^u e^{-r^2} dr$. We then have

$$(3.4) \quad \begin{aligned} \frac{1}{4\sqrt{\pi s}} &\leq (2\pi)^{-2} \int_{\mathbb{R}^2} e^{-\frac{s}{4}(-2+|\xi|^2)^2} d\xi = \frac{1 + \psi(\sqrt{s})}{4\sqrt{\pi}} \frac{1}{\sqrt{s}} \\ &= [1 + \psi(\sqrt{s})] \int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{s;\xi}^{\text{SFO}^d}|^2 d\xi \leq \frac{1}{2\sqrt{\pi s}} \end{aligned}$$

and the assertions in (3.1) and its immediate consequence (3.2) are established for $d = 2$.

For dimensions $d = 1, 3$, we get the desired estimates by comparing $\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}|^2 d\xi$ with $\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}|^2 d\xi$ (see (2.10) and (2.11) above). To start, we use (2.10) and observe that

$$(3.5) \quad \lim_{t \searrow 0} \frac{\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}|^2 d\xi}{\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}|^2 d\xi} = \lim_{t \searrow 0} \frac{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} d\xi}{\int_{\mathbb{R}^d} e^{-\frac{t}{4}|\xi|^4} d\xi} = 1; \quad d = 1, 2, 3.$$

From (2.10), (2.11), and (3.5), we then easily have

$$(3.6) \quad \begin{aligned} C_{\min}^{(d)} &:= \inf_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}|^2 d\xi}{\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}|^2 d\xi} = \inf_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} d\xi}{\int_{\mathbb{R}^d} e^{-\frac{t}{4}|\xi|^4} d\xi} > 0 \text{ and} \\ C_{\max}^{(d)} &:= \sup_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}|^2 d\xi}{\int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}|^2 d\xi} = \sup_{0 < t \leq T} \frac{\int_{\mathbb{R}^d} e^{-\frac{t}{4}(-2+|\xi|^2)^2} d\xi}{\int_{\mathbb{R}^d} e^{-\frac{t}{4}|\xi|^4} d\xi} < \infty, \end{aligned}$$

for $d = 1, 2, 3$. So, for $d = 1, 3$, and $0 < s \leq T$, we use the Parseval-Plancherel theorem together with (3.6) and (2.11) to get the desired lower and upper bounds as follows:

$$(3.7) \quad \begin{aligned} \int_{\mathbb{R}^d} |\mathbb{K}_{s;x}^{\text{LKS}^d}|^2 dx &= \int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{s;\xi}^{\text{LKS}^d}|^2 d\xi \geq C_{\min}^{(d)} \int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{s;\xi}^{\text{SFO}^d}|^2 d\xi = C_{\min}^{(d)} C_d s^{-\frac{d}{4}}, \\ \int_{\mathbb{R}^d} |\mathbb{K}_{s;x}^{\text{LKS}^d}|^2 dx &= \int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{s;\xi}^{\text{LKS}^d}|^2 d\xi \leq C_{\max}^{(d)} \int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{s;\xi}^{\text{SFO}^d}|^2 d\xi = C_{\max}^{(d)} C_d s^{-\frac{d}{4}}. \end{aligned}$$

The assertions in (3.1) and its immediate consequence (3.2) are thus established for $d = 1, 3$ and the proof is complete. \square

Remark 3.1. In $d = 1$, there is a critical $t_c > 1$ such that³⁵

$$(3.8) \quad \begin{cases} \int_{\mathbb{R}} |\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}|^2 d\xi < \int_{\mathbb{R}} |\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}|^2 d\xi, & t < t_c \\ \int_{\mathbb{R}} |\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}|^2 d\xi \geq \int_{\mathbb{R}} |\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}|^2 d\xi, & t \geq t_c, \end{cases}$$

with equality at $t = t_c$. If $T \leq t_c$, then using (2.11) and (3.8) the lower bound of (3.1) immediately holds with $C_l^{(1)} = C_1 = 1/2\Gamma(\frac{3}{4})$, where C_1 is the constant in (2.11) for $d = 1$. On the other hand, as in the case $d = 2$ (see (3.4) above), when $d = 3$ we have

$$(3.9) \quad \int_{\mathbb{R}^3} |\hat{\mathbb{K}}_{t;\xi}^{\text{SFO}^d}|^2 d\xi < \int_{\mathbb{R}^3} |\hat{\mathbb{K}}_{t;\xi}^{\text{LKS}^d}|^2 d\xi; \quad t > 0,$$

which, when combined with (2.11), gives us the lower bound with the constant $C_l^{(3)} = C_3 = \Gamma(\frac{3}{4})/\pi^2\sqrt{8}$.

Lemma 3.2 (Kernel's L^2 temporal difference). *Fix any arbitrary $T > 0$. There are constants $\tilde{C}_u^{(d)}$, depending only on d and T such that*

$$(3.10) \quad \int_0^t \int_{\mathbb{R}^d} |\mathbb{K}_{t-s;x}^{\text{LKS}^d} - \mathbb{K}_{r-s;x}^{\text{LKS}^d}|^2 dx ds \leq \tilde{C}_u^{(d)}(t-r)^{\frac{4-d}{4}}; \quad 0 < r < t \leq T, d = 1, 2, 3,$$

with the convention that $\mathbb{K}_{t;x}^{\text{LKS}^d} = 0$ if $t < 0$. The same estimate holds, with possibly different constants, when replacing $\mathbb{K}_{t;x}^{\text{LKS}^d}$ with $\mathbb{K}_{t;x}^{\text{SFO}^d}$.

Proof. Throughout the proof, unless otherwise specified, the spatial dimension $d \in \{1, 2, 3\}$. For $u, v > 0$ let

$$(3.11) \quad \tilde{\mathbb{K}}_{u+v}^{(d)} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{u}{8}(-2+|\xi|^2)^2} e^{-\frac{v}{8}(-2+|\xi|^2)^2} d\xi.$$

By the Parseval-Plancherel theorem, we have

$$(3.12) \quad \begin{aligned} & \int_0^t \int_{\mathbb{R}^d} |\mathbb{K}_{s+(t-r);x}^{\text{LKS}^d} - \mathbb{K}_{s;x}^{\text{LKS}^d}|^2 dx ds = \int_0^t \int_{\mathbb{R}^d} |\hat{\mathbb{K}}_{s+(t-r);\xi}^{\text{LKS}^d} - \hat{\mathbb{K}}_{s;\xi}^{\text{LKS}^d}|^2 d\xi ds \\ &= \int_0^t \left[\tilde{\mathbb{K}}_{2[s+(t-r)]}^{(d)} - 2\tilde{\mathbb{K}}_{2s+(t-r)}^{(d)} + \tilde{\mathbb{K}}_{2s}^{(d)} \right] ds \\ &= \left[\int_0^{\frac{t-r}{2}} \tilde{\mathbb{K}}_{2s}^{(d)} ds - \int_{\frac{t-r}{2}}^{t-r} \tilde{\mathbb{K}}_{2s}^{(d)} ds - \int_t^{t+\frac{t-r}{2}} \tilde{\mathbb{K}}_{2s}^{(d)} ds + \int_{t+\frac{t-r}{2}}^{2t-r} \tilde{\mathbb{K}}_{2s}^{(d)} ds \right]. \end{aligned}$$

³⁵ $t_c \approx 1.506188$. It is interesting to note that this is only a one-dimensional phenomenon (see (3.4) and (3.9)).

It is clear from (3.11) that $\tilde{\mathbb{K}}_{2s}^{(d)}$ is decreasing in s . Thus, the sum of the last three terms of (3.12) is ≤ 0 and we have

$$(3.13) \quad \begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \left| \mathbb{K}_{s+(t-r);x}^{\text{LKS}^d} - \mathbb{K}_{s;x}^{\text{LKS}^d} \right|^2 dx ds \leq \int_0^{\frac{t-r}{2}} \tilde{\mathbb{K}}_{2s}^{(d)} ds \\ & = \int_0^{\frac{t-r}{2}} \int_{\mathbb{R}^d} \left| \mathbb{K}_{s;x}^{\text{LKS}^d} \right|^2 dx ds \leq \tilde{C}_u^{(d)}(t-r)^{\frac{4-d}{4}}; 0 < r < t \leq T, d \in \{1, 2, 3\}, \end{aligned}$$

where we used the definition of $\tilde{\mathbb{K}}_{2s}^{(d)}$ in (3.11), Parseval-Plancherel theorem, and Lemma 3.1. The proof of the simpler $\mathbb{K}_{t;x}^{\text{SFO}^d}$ case follows the same steps, with obvious trivial changes, and will be omitted. The lemma is established³⁶. \square

Lemma 3.3 (Kernel's L^2 spatial difference). *For $d \in \{1, 2, 3\}$, there are intervals $I_1 = (0, 1]$, $I_2 = (0, 1)$, and $I_3 = (0, 1/2)$; positive numbers $\{\alpha_d \in I_d\}_{d=1}^3$; constants $\{C_u^{(d)}\}_{d=1}^3$ depending only on d and $\alpha_d \in I_d$ such that*

$$(3.14) \quad \int_0^t \int_{\mathbb{R}^d} \left| \mathbb{K}_{s;x}^{\text{LKS}^d} - \mathbb{K}_{s;x+z}^{\text{LKS}^d} \right|^2 dx ds \leq C_u^{(d)} |z|^{2\alpha_d} (1 \vee \frac{t}{4}); \forall \alpha_d \in I_d, t > 0,$$

where $0 < C_u^{(d)} < \infty$ for every $\alpha_d \in I_d$ for $d = 1, 2, 3$. The same estimate holds, with possibly different constants, when replacing $\mathbb{K}_{t;x}^{\text{LKS}^d}$ with $\mathbb{K}_{t;x}^{\text{SFO}^d}$.

Proof. We first observe from (2.4) that

$$(3.15) \quad \hat{\mathbb{K}}_{s;\xi+z}^{\text{LKS}^d} = (2\pi)^{-\frac{d}{2}} e^{-\frac{z}{4}(-2+|\xi|^2)^2} e^{iz \cdot \xi}.$$

Suppose $d \in \{1, 2, 3\}$, and let $\mathbb{B}_{\sqrt{2}}^d := \{\xi \in \mathbb{R}^d; |\xi| < \sqrt{2}\}$. Again, the Parseval-Plancherel theorem tells us that the quantity we want to estimate is

$$(3.16) \quad \begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \left| \hat{\mathbb{K}}_{s;\xi}^{\text{LKS}^d} - \hat{\mathbb{K}}_{s;\xi+z}^{\text{LKS}^d} \right|^2 d\xi ds \\ & = (2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} \left| e^{-\frac{s}{4}(-2+|\xi|^2)^2} [1 - e^{iz \cdot \xi}] \right|^2 d\xi ds \\ & = 2(2\pi)^{-d} \int_0^t \int_{\mathbb{R}^d} e^{-\frac{s}{4}(-2+|\xi|^2)^2} [1 - \cos(z \cdot \xi)] d\xi ds \\ & = 8(2\pi)^{-d} \int_{\mathbb{B}_{\sqrt{2}}^d} \left[\frac{1 - e^{-\frac{t}{4}(-2+|\xi|^2)^2}}{(-2+|\xi|^2)^2} \right] [1 - \cos(z \cdot \xi)] d\xi \\ & \quad + 8(2\pi)^{-d} \int_{\mathbb{R}^d \setminus (\mathbb{B}_{\sqrt{2}}^d \cup \partial \mathbb{B}_{\sqrt{2}}^d)} \left[\frac{1 - e^{-\frac{t}{4}(-2+|\xi|^2)^2}}{(-2+|\xi|^2)^2} \right] [1 - \cos(z \cdot \xi)] d\xi. \end{aligned}$$

³⁶The constants $\tilde{C}_u^{(d)} = \left[2^{\frac{d-4}{4}} \right] C_u^{(d)}$, where the constants $C_u^{(d)}$ are those in Lemma 3.1.

We make use of the following two sets of elementary inequalities for all $d \geq 1$, the first of which uses the Cauchy-Schwarz inequality to obtain the last bound

$$(3.17) \quad \begin{aligned} 1 - \cos(z \cdot \xi) &\leq 2 \left(1 \wedge |z \cdot \xi|^{2\alpha}\right) \leq 2 \left(1 \wedge |z|^{2\alpha} |\xi|^{2\alpha}\right); \quad 0 < \alpha \leq 1, \\ \frac{1 - e^{-\frac{t}{4}(-2+|\xi|^2)^2}}{(-2+|\xi|^2)^2} &\leq (1 \vee \frac{t}{4}) \wedge \frac{(1 \vee \frac{t}{4}) \left(1 - e^{-(2+|\xi|^2)^2}\right)}{(-2+|\xi|^2)^2}; \quad t \geq 0. \end{aligned}$$

We now treat the cases $d = 1, 2, 3$ separately. Using (3.16), (3.17), and changing to polar coordinates in $d = 2$ and to spherical coordinates in $d = 3$ we can bound our desired quantity

$$\int_0^t \int_{\mathbb{R}^d} \left| \hat{\mathbb{K}}_{s;\xi}^{\text{LKS}^d} - \hat{\mathbb{K}}_{s;\xi+z}^{\text{LKS}^d} \right|^2 d\xi ds$$

from above by

$$(3.18) \quad \begin{aligned} &\frac{16(1 \vee \frac{t}{4})}{2\pi} |z|^{2\alpha} \left[\int_{-\sqrt{2}}^{\sqrt{2}} |\xi|^{2\alpha} d\xi + 2 \int_{\sqrt{2}}^{\infty} \frac{1 - e^{-(2+|\xi|^2)^2}}{(-2+|\xi|^2)^2} |\xi|^{2\alpha} d\xi \right] \\ &\leq C^{(1)} |z|^{2\alpha}; \quad 0 < \alpha \leq 1 \text{ for } d = 1, \end{aligned}$$

$$(3.19) \quad \begin{aligned} &\frac{16(1 \vee \frac{t}{4})}{(2\pi)^2} |z|^{2\alpha} \left[\int_0^{2\pi} \int_0^{\sqrt{2}} r^{2\alpha} r dr d\theta + \int_0^{2\pi} \int_{\sqrt{2}}^{\infty} \frac{1 - e^{-(2+r^2)^2}}{(-2+r^2)^2} r^{2\alpha} r dr d\theta \right] \\ &\leq C^{(2)} |z|^{2\alpha}; \quad 0 < \alpha < 1 \text{ for } d = 2, \end{aligned}$$

and

$$(3.20) \quad \begin{aligned} &\frac{16(1 \vee \frac{t}{4})}{(2\pi)^3} |z|^{2\alpha} \left[\int_0^{\pi} \int_0^{2\pi} \int_0^{\sqrt{2}} r^{2\alpha} r^2 \sin(\vartheta) dr d\theta d\vartheta \right. \\ &\quad \left. + \int_0^{\pi} \int_0^{2\pi} \int_{\sqrt{2}}^{\infty} \frac{1 - e^{-(2+r^2)^2}}{(-2+r^2)^2} r^{2\alpha} r^2 \sin(\vartheta) dr d\theta d\vartheta \right] \\ &\leq C^{(3)} |z|^{2\alpha}; \quad 0 < \alpha < \frac{1}{2} \text{ for } d = 3. \end{aligned}$$

In particular, when $d = 1$, α may be taken to be 1; in $d = 2$, $\alpha \in (0, 1)$; and in $d = 3$, $\alpha \in (0, 1/2)$. Our dimension-dependent upper bound constant $C^{(d)}$ is independent of t if $t \leq 4$ and increases with t if $t > 4$. The proof of the simpler $\mathbb{K}_{t;x}^{\text{SFO}^d}$ case follows the same steps, with obvious trivial changes, and will be omitted. \square

3.2. Finishing the proof of Theorem 1.1. We now complete the proof of Theorem 1.1 by first establishing the Hölder regularity results without imposing any Lipschitz conditions on a , assuming the L^p boundedness of solutions on $\mathbb{T} \times \mathbb{R}^d$. We then add a Lipschitz condition on a and obtain the strong (stochastically) existence and uniqueness result for the L-KS SPDE (1.12), together with the L^p boundedness assumed before; we thus obtain the Hölder regularity with no L^p boundedness assumptions³⁷. With Lemma 3.1–Lemma 3.3 in hand, the rest of the proof of Theorem 1.1 is a straightforward adaptation of our corresponding arguments in [2] to our

³⁷As we mentioned in Remark 1.2 the existence of lattice limit solutions along with the regularity results in Theorem 1.1 (including both L^p boundedness on $\mathbb{T} \times \mathbb{R}^d$ and Hölder regularity) can be proven under the weaker non-Lipschitz conditions (NLip), as we did in [1, 2].

setting here. For the convenience of the reader; we state the needed results below, referring to [2] for the details and noting the minor notational changes. Without loss of generality, assume for the remainder of the section that $(\varepsilon, \vartheta) = (1, 1)$ and that U solves the L-KS SPDE (1.12).

Recalling that the initial data u_0 is assumed deterministic and writing U in the kernel formulation (1.11) in terms of its deterministic and random parts $U(t, x) = U_D(t, x) + U_R(t, x)$, we note that the deterministic part $U_D(t, x) = \int_{\mathbb{R}^d} \mathbb{K}_{t;x,y}^{\text{LKS}^d} u_0(y) dy$ is $C^{1,4}(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R})$ smooth in time and space, under the assumptions on u_0 , since it is a classical solution to the LKS PDE (1.12) for $(\varepsilon, \vartheta) = (1, 1)$. So, it suffices to get the Hölder regularity of the random part. We now give estimates on the spatial and temporal differences of the random part U_R . To get straight to these important regularity estimates, we first assume that

$$(3.21) \quad M_q(t) = \sup_{x \in \mathbb{R}^d} \mathbb{E}|U(t, x)|^{2q} \leq K_{T,q} < \infty; \quad t \in \mathbb{T} = [0, T], q \geq 1.$$

Lemma 3.4 (Spatial and temporal differences L^{2q} estimates). *Assume that (Lip) and (3.21) are in force. There exists a constant \tilde{C}_d depending only on $q, \max_x |u_0(x)|$, the spatial dimension $d \in \{1, 2, 3\}$, α_d , and T such that*

$$\mathbb{E}|U_R(t, x) - U_R(t, y)|^{2q} \leq \tilde{C}_d |x - y|^{2q\alpha_d}; \quad \alpha_d \in I_d,$$

for all $x, y \in \mathbb{R}^d$, $t \in \mathbb{T}$, and $d \in \{1, 2, 3\}$; where α_d and I_d are as in Lemma 3.3.

Also, there exists a constant \bar{C}_d depending only on $q, \max_x |u_0(x)|$, the spatial dimension $d \in \{1, 2, 3\}$, and T such that

$$\mathbb{E}|U_R(t, x) - U_R(r, x)|^{2q} \leq \bar{C}_d |t - r|^{\frac{(4-d)q}{4}},$$

for all $x \in \mathbb{R}^d$, for all $t, r \in \mathbb{T}$, and for $1 \leq d \leq 3$.

Proof. The proof follows the same exact steps as in the proofs of Lemma 2.5 and Lemma 2.6 in [2], replacing the kernel there with the L-KS kernel here and using Lemma 3.1–Lemma 3.3 here in place of the corresponding ones there. \square

We now have the desired Hölder regularity result as the following corollary.

Corollary 3.1 (Hölder regularity). *Assume that (U, \mathcal{W}) is an L-KS kernel solution to (1.12) on $\{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3$. Suppose further that the L^p boundedness in (3.21) holds. Then*

$$U \in H^{\frac{4-d}{8}, \left(\frac{4-d}{2} \wedge 1\right)^-}(\mathbb{T} \times \mathbb{R}^d; \mathbb{R}); \quad \text{for } d = 1, 2, 3.$$

almost surely.

Proof. It suffices to prove it for the random part. By Lemma 3.4 we easily have

$$(3.22) \quad \begin{cases} \mathbb{E}|U_R(t, x) - U_R(t, y)|^{2n+2d} \leq C_d |x - y|^{(2n+2d)\alpha_d}, \\ \mathbb{E}|U_R(t, x) - U_R(r, x)|^{2m+4d} \leq \bar{C}_d |t - r|^{\frac{(4-d)(m+2d)}{4}}, \end{cases}$$

for $d = 1, 2, 3$. Thus, by standard results, the spatial Hölder exponent is $\gamma_s \in \left(0, \frac{2(n+d)\alpha_d-d}{2n+2d}\right)$ and the temporal exponent is $\gamma_t \in \left(0, \frac{m(1-d/4)+d(1-d/2)}{2m+4d}\right) \forall m, n$.

Taking the limits as $m, n \rightarrow \infty$, we get $\gamma_t \in (0, \frac{4-d}{8})$ and $\gamma_s \in (0, \alpha_d)$, for $1 \leq d \leq 3$ and $\alpha \in I_d$ as in Lemma 3.3. The proof is now complete. \square

The final piece needed for the proof of Theorem 1.1 is now given by the following Lemma, which also removes the L^p boundedness assumption (3.21) by asserting that it automatically holds under the conditions (Lip).

Lemma 3.5 (Existence, uniqueness, and L^p boundedness). *Under the conditions in (Lip), there exists a strong and pathwise unique solution to the L-KS SPDE (1.12) on $\mathbb{R}_+ \times \mathbb{R}^d$ that is $L^p(\Omega)$ -bounded on $\mathbb{T} \times \mathbb{R}^d$, for every $p \geq 2$ and every $d = 1, 2, 3$.*

Proof. The proof follows exactly the same steps as the proof on pp. 27–29 in [2] for the BTBM SIE, with now obvious and minor changes from the BTBM setting of [2] to our L-KS setting here, we omit the details and point the interested reader to [2] for the specifics. The proof of Theorem 1.1 is now complete. \square

4. PROOF OF THEOREM 1.2

We now turn to the proof of Theorem 1.2. Again, without loss of generality, it is enough to fix³⁸ $\vartheta = 1$. We first need the ε_1 -time-scaled L-KS Kernel $\mathbb{K}_{\varepsilon_1 t; x}^{\text{LKS}^d}$, which—upon taking $\vartheta = 1$ in (1.10)—reduces to

$$\begin{aligned} \mathbb{K}_{\varepsilon_1 t; x}^{\text{LKS}^d} &= \mathbb{K}_{t; x}^{(\varepsilon_1, 1)\text{LKS}^d} = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon_1 t}{8}(-2+|\xi|^2)^2} e^{i\xi \cdot x} d\xi, \\ (4.1) \qquad \qquad \qquad &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-\frac{\varepsilon_1 t}{8}(-2+|\xi|^2)^2} \cos(\xi \cdot x) d\xi \end{aligned}$$

and which, by Theorem 2.1, solves the L-KS PDE

$$(4.2) \quad \begin{cases} \frac{\partial u}{\partial t} = -\frac{\varepsilon_1}{8} \Delta^2 u - \frac{\varepsilon_1}{2} \Delta u - \frac{\varepsilon_1}{2} u, & (t, x) \in (0, +\infty) \times \mathbb{R}^d; \\ u(0, x) = \delta(x); & x \in \mathbb{R}^d. \end{cases}$$

Also, exactly as in Lemma 3.1 above, $\mathbb{K}_{\varepsilon_1 t; x}^{\text{LKS}^d}$ satisfies the bounds³⁹.

$$(4.3) \quad \frac{C_u^{(d)} t^{\frac{4-d}{4}}}{\varepsilon_1^{d/4}} \geq \int_0^t \int_{\mathbb{R}^d} \left| \mathbb{K}_{\varepsilon_1 s; x}^{\text{LKS}^d} \right|^2 dx ds \geq \frac{C_l^{(d)} t^{\frac{4-d}{4}}}{\varepsilon_1^{d/4}}$$

Proof of Theorem 1.2. Let $T > 0$ and $q \geq 1$ be fixed and arbitrary, and let $(t, x) \in [0, T] \times \mathbb{R}^d$ and $d = 1, 2, 3$.

³⁸Again, Theorem 1.2 holds for any fixed value of $\vartheta \in \mathbb{R}$. Fixing ϑ value to 1 is for convenience and for simplifying notation and exposition in the proof only. The case $\vartheta = 1$ captures the general case of the 4th order biLaplacian term together with the second and zero order terms.

³⁹Of course, the case $\vartheta = 0$ satisfies (4.2), with only the biLaplacian term, without the Laplacian and without the zero order terms; and satisfies (4.3) with equality.

- (i) Under the conditions (Lip), we have by Theorem 1.1 a unique solution $U_{\varepsilon_1, \varepsilon_2}$ to the L-KS SPDE (1.17) that is $L^{2q}(\Omega)$ -bounded on $\mathbb{T} \times \mathbb{R}^d$, for every $q \geq 1$. Let $\mu_{\varepsilon_1}^{t,x}$ be the measure on $[0, t] \times \mathbb{R}^d$ defined by

$$d\mu_{\varepsilon_1}^{t,x}(s, y) = \left| \mathbb{K}_{\varepsilon_1(t-s); x, y}^{\text{LKS}^d} \right|^2 ds dy$$

and let $|\mu_{\varepsilon_1}^{t,x}| = \mu_{\varepsilon_1}^{t,x}([0, t] \times \mathbb{R})$. Taking the $2q$ -th moment of the difference between our scaled L-KS SPDE and its deterministic counterpart—whose solution we denote by u_{ε_1} ; using Burkholder's inequality followed by Jensen's inequality applied to the probability measure $d\mu_{\varepsilon_1}^{t,x}(s, y)/|\mu_{\varepsilon_1}^{t,x}|$; then using the linear growth condition ((b) in (Lip)) on a , the $L^{2q}(\Omega)$ -boundedness of $U_{\varepsilon_1, \varepsilon_2}$ on $\mathbb{T} \times \mathbb{R}^d$, and the upper bound in (4.3), we get

$$\begin{aligned}
 & \mathbb{E} |U_{\varepsilon_1, \varepsilon_2}(t, x) - u_{\varepsilon_1}(t, x)|^{2q} \\
 & \leq C \mathbb{E} \left| \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{\varepsilon_1(t-s); x, y}^{\text{LKS}^d} \varepsilon_2 a(U_{\varepsilon_1, \varepsilon_2}(s, y)) \mathscr{W}(ds, dy) \right|^{2q} \\
 (4.4) \quad & \leq C \varepsilon_2^{2q} \int_{\mathbb{R}^d} \int_0^t \mathbb{E} a^{2q}(U_{\varepsilon_1, \varepsilon_2}(s, y)) \frac{d\mu_{\varepsilon_1}^{t,x}(s, y)}{|\mu_{\varepsilon_1}^{t,x}|} |\mu_{\varepsilon_1}^{t,x}|^q \\
 & \leq \frac{CT^{\frac{(4-d)q}{4}} \varepsilon_2^{2q}}{\varepsilon_1^{(d/4)q}} \rightarrow 0
 \end{aligned}$$

as $\varepsilon_1, \varepsilon_2$, and $\varepsilon_2/\varepsilon_1^{d/8}$ approach 0.

- (ii) We prove it by contradiction. So, assume there is a $T > 0$ such that

$$(4.5) \quad \lim_{\substack{\varepsilon_1, \varepsilon_2 \downarrow 0 \\ \varepsilon_2/\varepsilon_1^{d/8} \rightarrow \infty}} \sup_{0 \leq s \leq T} \sup_{x \in \mathbb{R}^d} \mathbb{E} U_{\varepsilon_1, \varepsilon_2}^2(s, x) < \infty; \quad d = 1, 2, 3$$

and assume without loss of generality that $u_0 \equiv 0$. Observe that

$$\begin{aligned}
 & \mathbb{E} |U_{\varepsilon_1, \varepsilon_2}(t, x)|^2 = \mathbb{E} \left| \int_{\mathbb{R}^d} \int_0^t \mathbb{K}_{\varepsilon_1(t-s); x, y}^{\text{LKS}^d} \varepsilon_2 a(U_{\varepsilon_1, \varepsilon_2}(s, y)) \mathscr{W}(ds, dy) \right|^2 \\
 (4.6) \quad & = \varepsilon_2^2 \int_{\mathbb{R}^d} \int_0^t \left| \mathbb{K}_{\varepsilon_1(t-s); x, y}^{\text{LKS}^d} \right|^2 \mathbb{E} a^2(U_{\varepsilon_1, \varepsilon_2}(s, y)) ds dy \\
 & \geq K_l^2 \varepsilon_2^2 \int_{\mathbb{R}^d} \int_0^t \left| \mathbb{K}_{\varepsilon_1(t-s); x, y}^{\text{LKS}^d} \right|^2 ds dy \geq \frac{\tilde{C}^{(d)} \varepsilon_2^2 t^{\frac{4-d}{4}}}{\varepsilon_1^{d/4}}; \quad d = 1, 2, 3,
 \end{aligned}$$

where we used the lower bound assumption $0 < K_l \leq a(u)$ and the lower bound in (4.3) to get the last two inequalities in (4.6). Using the assumption in (4.5), we arrive at the desired contradiction by taking the limit as $\varepsilon_1, \varepsilon_2 \searrow 0$ in (4.6) such that $\varepsilon_2/\varepsilon_1^{d/8} \nearrow \infty$ for $d = 1, 2, 3$.

The proof is complete. \square

5. PROOF OF THEOREM 1.3

Here, we prove the change of measure transfer of properties from the canonical L-KS to nonlinear L-KS fourth order SPDEs, including the Swift-Hohenberg SPDE on subsets of $\{\mathbb{R}_+ \times \mathbb{R}^d\}_{d=1}^3$.

We say that a progressively measurable random field X on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfies Novikov's condition on $\mathbb{T} \times \mathbb{S}$ if

$$(5.1) \quad \mathbb{E}_{\mathbb{P}} \left[\exp \left(\frac{1}{2} \int_{\mathbb{T} \times \mathbb{S}} X^2(t, x) dt dx \right) \right] < \infty.$$

Proof of Theorem 1.3.

- (i) (Transfer of law uniqueness) We prove the more interesting direction (from zero to nonzero drift). The proof of the reverse direction is similar and is omitted. Suppose that uniqueness in law holds for the zero-drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ and that $(V^{(i)}, \tilde{\mathcal{W}}^{(i)})$, $(\Omega^{(i)}, \mathcal{F}^{(i)}, \{\mathcal{F}_t^{(i)}\}, \tilde{\mathbb{P}}^{(i)})$; $i = 1, 2$ are solutions to the nonzero-drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$. By assumption, we have

$$(5.2) \quad \tilde{\mathbb{P}}^{(i)} \left[\int_{\mathbb{T} \times \mathbb{S}} R_{V^{(i)}}^2(t, x) dt dx < \infty \right] = 1, \quad i = 1, 2.$$

Define the sequence of stopping times $\{\tau_n^{(i)}\}$ by

$$(5.3) \quad \tau_n^{(i)} := T \wedge \inf \left\{ 0 \leq t \leq T; \int_{[0, t] \times \mathbb{S}} R_{V^{(i)}}^2(s, x) ds dx = n \right\}; \quad n \in \mathbb{N}, \quad i = 1, 2,$$

and let $\mathcal{W}^{(i)} = \{\mathcal{W}_t^{(i)}(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{S})\}$ be given by

$$\mathcal{W}_t^{(i)}(B) := \tilde{\mathcal{W}}_t^{(i)}(B) + \int_{[0, t] \times B} R_{V^{(i)}}(s, x) ds dx; \quad i = 1, 2.$$

Then, Novikov's condition (5.1) and Girsanov's theorem for white noise (see Theorem 2.2, Corollary 2.3, and Lemma 2.4 in [13]) immediately gives us that $\mathcal{W}_n^{(i)} = \{\mathcal{W}_{t \wedge \tau_n^{(i)}}^{(i)}(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{S})\}$ is a white noise stopped at time $\tau_n^{(i)}$, under the probability measure $\mathbb{P}_n^{(i)}$ defined on $\mathcal{F}_T^{(i)}$ by

$$\frac{d\mathbb{P}_n^{(i)}}{d\tilde{\mathbb{P}}^{(i)}} = \Upsilon_{T \wedge \tau_n^{(i)}}^{R_{V^{(i)}}, \tilde{\mathcal{W}}^{(i)}}(\mathbb{S}); \quad n \in \mathbb{N}, \quad i = 1, 2,$$

where the Radon-Nikodym derivative is given by

$$\begin{aligned} & \Upsilon_{t \wedge \tau_n^{(i)}}^{R_{V^{(i)}}, \tilde{\mathcal{W}}^{(i)}}(B) \\ &:= \exp \left\{ \int_{[0, t \wedge \tau_n^{(i)}] \times B} - \left[R_{V^{(i)}}(s, x) \tilde{\mathcal{W}}^{(i)}(ds, dx) - \frac{1}{2} R_{V^{(i)}}^2(s, x) ds dx \right] \right\}; \end{aligned}$$

$0 \leq t \leq T$, $B \in \mathcal{B}(\mathbb{S})$. Consequently, $(V^{(i)}, \mathcal{W}_n^{(i)})$, $(\Omega^{(i)}, \mathcal{F}_T^{(i)}, \{\mathcal{F}_t^{(i)}\}, \mathbb{P}_n^{(i)})$ is a solution to the zero-drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ on $[0, T \wedge \tau_n^{(i)}] \times \mathbb{S}$

for each $i = 1, 2$ and $n \in \mathbb{N}$. Clearly, for $i = 1, 2$,

$$(5.4) \quad \begin{aligned} \frac{d\tilde{\mathbb{P}}^{(i)}}{d\mathbb{P}_n^{(i)}} &= \Xi_{T \wedge \tau_n^{(i)}}^{R_{V^{(i)}}, \mathcal{W}^{(i)}}(\mathbb{S}) \\ &:= \exp \left\{ \int_{[0, T \wedge \tau_n^{(i)}] \times \mathbb{S}} \left[R_{V^{(i)}}(s, x) \mathcal{W}^{(i)}(ds, dx) - \frac{1}{2} R_{V^{(i)}}^2(s, x) ds dx \right] \right\}; \end{aligned}$$

$n \in \mathbb{N}$. Thus, for any set $\Lambda \in \mathcal{B}(C(\mathbb{T} \times \mathbb{S}; \mathbb{R}))$

$$(5.5) \quad \begin{aligned} \tilde{\mathbb{P}}^{(1)} \left[V^{(1)} \in \Lambda, \tau_n^{(1)} = T \right] &= \mathbb{E}_{\mathbb{P}_n^{(1)}} \left[1_{\{V^{(1)} \in \Lambda, \tau_n^{(1)} = T\}} \Xi_{T \wedge \tau_n^{(1)}}^{R_{V^{(1)}}, \mathcal{W}^{(1)}}(\mathbb{S}) \right] \\ &= \mathbb{E}_{\mathbb{P}_n^{(2)}} \left[1_{\{V^{(2)} \in \Lambda, \tau_n^{(2)} = T\}} \Xi_{T \wedge \tau_n^{(2)}}^{R_{V^{(2)}}, \mathcal{W}^{(2)}}(\mathbb{S}) \right] \\ &= \tilde{\mathbb{P}}^{(2)} \left[V^{(2)} \in \Lambda, \tau_n^{(2)} = T \right]; \quad \forall n \in \mathbb{N}, \end{aligned}$$

where we have used the uniqueness in law assumption on $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ (comparing the $V^{(i)}$'s only on $\Omega_n^{(i)} := \{\tau_n^{(i)} = T\}$ for each n), (5.3), and (5.4) to get the second equality in (5.5). By (5.2) and (5.3), we obtain that $\lim_{n \rightarrow \infty} \tilde{\mathbb{P}}^{(i)}[\tau_n^{(i)} = T] = 1$ for $i = 1, 2$. Thus, taking the limit as $n \rightarrow \infty$ in (5.5) yields that the law of $V^{(1)}$ under $\tilde{\mathbb{P}}^{(1)}$ is the same as that of $V^{(2)}$ under $\tilde{\mathbb{P}}^{(2)}$. I.e., we have uniqueness in law for the non-zero drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$.

- (ii) (Law equivalence) Let $(V, \mathcal{W}^{(1)})$ be a solution (weak or strong) to the nonzero-drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ on $(\Omega^{(1)}, \mathcal{H}, \{\mathcal{H}_t\}, \mathbb{Q})$; and let $(U, \mathcal{W}^{(2)})$ be a solution (weak or strong) to the zero-drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ on $(\Omega^{(2)}, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Then, uniqueness in law for the L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ follows from the uniqueness in law assumption for $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$, the almost sure $L^2(\mathbb{T} \times \mathbb{S}, \lambda)$ condition on R_V , and part (i) of Theorem 1.3.

Replacing $V^{(i)}$ in (5.3) by U and then V , we get the definitions of the stopping times sequences $\{\tau_n^U\}$ and $\{\tau_n^V\}$, respectively. Let $\tilde{\mathcal{W}} = \{\tilde{\mathcal{W}}_t(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{S})\}$ be given by

$$\tilde{\mathcal{W}}_t(B) := \mathcal{W}_t^{(2)}(B) - \int_{[0, t] \times B} R_U(s, x) ds dx.$$

Then, Novikov's condition and Girsanov's theorem for white noise (see Theorem 2.2, Corollary 2.3, and Lemma 2.4 in [13]) immediately give us that, for $n \in \mathbb{N}$, $\tilde{\mathcal{W}}_n = \{\tilde{\mathcal{W}}_{t \wedge \tau_n^U}(B), \mathcal{F}_t; 0 \leq t \leq T, B \in \mathcal{B}(\mathbb{S})\}$ is a white noise stopped at time τ_n^U , under the probability measure $\tilde{\mathbb{P}}_n$ defined on \mathcal{F}_T by the Radon-Nikodym derivative

$$(5.6) \quad \begin{aligned} \frac{d\tilde{\mathbb{P}}_n}{d\mathbb{P}} &= \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}^{(2)}}(\mathbb{S}) \\ &:= \exp \left\{ \int_{[0, T \wedge \tau_n^U] \times \mathbb{S}} R_U(s, x) \mathcal{W}^{(2)}(ds, dx) - \frac{1}{2} \int_{[0, T \wedge \tau_n^U] \times \mathbb{S}} R_U^2(s, x) ds dx \right\}. \end{aligned}$$

Thus, $(U, \tilde{\mathcal{W}}_n), (\Omega^{(2)}, \mathcal{F}_T, \{\mathcal{F}_t\}, \tilde{\mathbb{P}}_n)$ is a solution to the nonzero-drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ on $[0, T \wedge \tau_n^U] \times \mathbb{S}$, for each $n \in \mathbb{N}$. As a result, for any set $\Lambda \in \mathcal{B}(C(\mathbb{T} \times \mathbb{S}; \mathbb{R}))$ we get

$$(5.7) \quad \begin{aligned} \mathbb{Q}[V \in \Lambda, \tau_n^V = T] &= \tilde{\mathbb{P}}_n[U \in \Lambda, \tau_n^U = T] \\ &= \mathbb{E}_{\mathbb{P}} \left[1_{\{U \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}}(\mathbb{S}) \right]; \quad n \in \mathbb{N}. \end{aligned}$$

To see (5.7) note that, on the event $\Omega_n^U := \{\omega \in \Omega^{(2)}; \tau_n^U(\omega) = T\}$, $(U, \tilde{\mathcal{W}}_n)$ is a solution to $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ on $\mathbb{T} \times \mathbb{S}$, under $\tilde{\mathbb{P}}_n$. Thus, the first equality in (5.7) follows from the uniqueness in law for $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ and the definitions of τ_n^U and τ_n^V . By the L^2 assumption on R_V and the definition of τ_n^V , we have $\lim_{n \rightarrow \infty} \mathbb{Q}[\tau_n^V = T] = 1$; so, taking limits in (5.7) we get

$$(5.8) \quad \mathbb{Q}[V \in \Lambda] = \lim_{n \rightarrow \infty} \tilde{\mathbb{P}}_n[U \in \Lambda, \tau_n^U = T] = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[1_{\{U \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}^{(2)}}(\mathbb{S}) \right].$$

Obviously, if $\mathbb{P}[U(\cdot, \cdot) \in \Lambda] = 0$ then $\mathbb{E}_{\mathbb{P}} \left[1_{\{U(\cdot, \cdot) \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}^{(2)}}(\mathbb{S}) \right] = 0$ for each n ; thus,

$$\mathbb{Q}[V(\cdot, \cdot) \in \Lambda] = \lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[1_{\{U(\cdot, \cdot) \in \Lambda, \tau_n^U = T\}} \Xi_{T \wedge \tau_n^U}^{R_U, \mathcal{W}^{(2)}}(\mathbb{S}) \right] = 0.$$

I.e., $\mathcal{L}_{\mathbb{Q}}^U$ is absolutely continuous with respect to $\mathcal{L}_{\mathbb{P}}^U$ on $\mathcal{B}(C(\mathbb{T} \times \mathbb{S}; \mathbb{R}))$. The absolute continuity of $\mathcal{L}_{\mathbb{P}}^U$ with respect to $\mathcal{L}_{\mathbb{Q}}^U$ is proved by a similar argument, and we omit it.

The proof of the last part of Theorem 1.3 follows since the square of the drift/diffusion ratios given by the random fields

$$(5.9) \quad R_U^2(t, x) = \sum_{k=0}^{2N} \tilde{c}_k U^k(t, x) \text{ and } R_V^2(t, x) = \sum_{k=0}^{2N} \tilde{c}_k V^k(t, x) \text{ for } \tilde{c}_i \in \mathbb{R}$$

are continuous and thus almost surely bounded on the compact set $[0, T] \times \mathbb{S}$. Thus, R_U and R_V are in $L^2(\mathbb{T} \times \mathbb{S}, \lambda)$, almost surely, and parts (i) and (ii) of Theorem 1.3 follow in the case $a(u) = \kappa \in \mathbb{R} \setminus \{0\}$, $b(u) = \sum_{k=0}^N c_k u^k$ for $c_k \in \mathbb{R}$, $k = 0, \dots, N$, and $N \geq 0$, and $u_0 \in C_c^{2, \gamma}(\mathbb{S}; \mathbb{R})$ and nonrandom.. The uniqueness assertion for $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, b, u_0)$ follows from (i) together with Lemma A.1 and the Hölder equivalence assertion follows from (ii). \square

APPENDIX A. UNIQUENESS LEMMA

Throughout this Appendix, we reserve the notation \mathbb{S} solely for the d -dimensional rectangle $\prod_{i=1}^d [0, L_i]$, $d = 1, 2, 3$; and, as before, $\mathbb{T} = [0, T]$ for some fixed but arbitrary $T > 0$. We now prove pathwise uniqueness for $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ with Dirichlet boundary conditions. The following lemma is useful for the last part of Theorem 1.3.

Lemma A.1. *Pathwise uniqueness, and hence uniqueness in law, holds for the zero-drift L-KS SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ on $\mathbb{R}_+ \times \mathbb{S}$ whenever $a \equiv \kappa \neq 0$.*

Proof. Assume without loss of generality that $d = 1$, $\mathbb{S} = [0, 1]$, and fix $\varepsilon > 0$, $\vartheta \in \mathbb{R}$. Assume further that $(U^{(1)}, \mathscr{W})$ and $(U^{(2)}, \mathscr{W})$ are two continuous solutions to $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ on $\mathbb{R}_+ \times \mathbb{S}$ on the same usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and with respect to the same space-time white noise \mathscr{W} . Let $D(t, x) = U^{(1)}(t, x) - U^{(2)}(t, x)$ for $(t, x) \in \mathbb{R}_+ \times \mathbb{S}$, and let

$$\Phi_{\text{Dir}}^\infty(\mathbb{S}; \mathbb{R}) := \{\varphi \in C^\infty(\mathbb{R}^d; \mathbb{R}); \varphi = \Delta\varphi = 0 \text{ on } \partial\mathbb{S}\}.$$

Now, the continuous difference random field D satisfies

$$\begin{aligned} (A.1) \quad & \int_0^1 D(t, x) \varphi(x) dx \\ &= -\frac{\varepsilon}{8} \int_0^t \int_0^1 D(s, x) \left(\varphi^{(4)}(x) + 4\vartheta \varphi^{(2)}(x) + 4\vartheta^2 \varphi(x) \right) dx ds \end{aligned}$$

for every $\varphi \in \Phi_{\text{Dir}}^\infty(\mathbb{S}; \mathbb{R})$, $t \in \mathbb{T}$, a.s. \mathbb{P} . Manifestly, this implies that $D(t, x) = 0$ on $[0, T] \times [0, 1]$ a.s. \mathbb{P} . To see this, choose $\varphi_m(x) = \sin(m\pi x)$ for $m \in \mathbb{N}$ and let

$$C_m(t) := \int_0^1 D(t, x) \varphi_m(x) dx; \quad \forall m \in \mathbb{N}, t \in \mathbb{T}.$$

So, by (A.1), we have

$$(A.2) \quad C_m(t) = \left(-\frac{\varepsilon m^4 \pi^4}{8} + \frac{\varepsilon \vartheta m^2 \pi^2}{2} - \frac{\varepsilon \vartheta^2}{2} \right) \int_0^t C_m(s) ds; \quad \text{a.s. } \mathbb{P},$$

for all $t \in [0, T]$ and $m \in \mathbb{N}$, which obviously implies that, for each m , $C_m(t) = 0$ for all $t \in \mathbb{T}$ a.s. \mathbb{P} . Now, since all the Fourier Sine coefficients, $C_m(t)$, for $D(t, x)$ are zero and the continuous solutions of the SPDE $e_{\text{LKS}}^{(\varepsilon, \vartheta)}(a, 0, u_0)$ vanish at $x = 0$ and $x = 1$ for all t (and hence $D(t, 0) = D(t, 1) = 0 \forall t$), we get that $D(t, x) = 0$ on $[0, T] \times [0, 1]$ a.s. \mathbb{P} . The arbitrariness of T now completes the proof. \square

APPENDIX B. FREQUENT ACRONYMS AND NOTATIONS KEY

I. Acronyms

- (1) BTBM: Brownian-time Brownian motion.
- (2) BTBM SIE: BTBM stochastic integral equation.
- (3) IBTBAP: imaginary Brownian-time Brownian angle process.
- (4) KS: Kuramoto Sivashinsky.
- (5) SH: Swift-Hohenberg.

II. Notations

- (1) $\mathbb{K}_{t;x}^{(\varepsilon, \vartheta)\text{LKS}^d}$ the (ε, ϑ) L-KS kernel (see (1.8)).
- (2) $\mathbb{K}_{t;x}^{\text{LKS}^d} = \mathbb{K}_{t;x}^{(1,1)\text{LKS}^d}$ the canonical (or $(1, 1)$) L-KS kernel (see (1.5) and (1.8)).
- (3) $\mathbb{K}_{t;x}^{\text{SFO}^d} := \mathbb{K}_{t;x}^{(1,0)\text{LKS}^d}$ the zero-angle canonical (or simple fourth order) kernel (see (2.8) and (1.8)).
- (4) $C^{k,\gamma}$ the set of functions with γ -Hölder continuous k -th derivative, $k = 0, 1, 2, \dots$

- (5) $C_b^{k,\gamma}$ the set of functions with γ -Hölder continuous and bounded k -th derivative, $k = 0, 1, 2, \dots$
- (6) $\mathbb{T} := [0, T]$ for some fixed and arbitrary $T > 0$
- (7) $M_p(t) := \sup_{x \in \mathbb{R}^d} \mathbb{E} |U(t, x)|^p$

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